# 复合材料层合板的几何非线性基本方程

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根据Reissner-Mindlin剪切变形理论，本文讨论了复合材料层合板的几何非线性力学问题。得到了一般层合板的控制方程，最后列出了七种典型的边界条件。

## 一 一般层合板的本构关系及控制方程

对于一任意层合板，取全板中面为坐标面xoy。我们假定板中面的法线变形后仍为直线，但不再垂直于变形后的中面。设$\psi_x$，$\psi_y$为中面法线变形后在xz和yz平面的转角。并约定从z轴正方向转动到x轴或y轴正方向为正。因此中面外任意一点的位移为：

$$\begin{aligned} u(x, y, z, t) &= u_0(x, y, t) + z\psi_x(x, y, t) \\ v(x, y, z, t) &= v_0(x, y, t) + z\psi_y(x, y, t) \\ w(x, y, z, t) &= w_0(x, y, t) \end{aligned} \tag{1}$$

$u_0$，$v_0$，$w_0$是板的中面位移。考虑横向剪切变形时层合板的本构关系为

$$\begin{pmatrix} N_i \\ M_i \end{pmatrix} = \begin{pmatrix} A_{ij} & B_{ij} \\ B_{ij} & D_{ij} \end{pmatrix} \begin{pmatrix} \varepsilon_j^0 \\ K_j \end{pmatrix} \tag{2}$$

这里的$N_i$和$M_i$是中面单位宽度的内力

$$[N_i] = [N_x, N_y, N_{xy}]^T = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} [\sigma_x, \sigma_y, \sigma_{xy}]^T dz$$

$$[M_i] = [M_x, M_y, M_{xy}]^T = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} [\sigma_x, \sigma_y, \sigma_{xy}]^T z dz \tag{3}$$

$\varepsilon^0_j$和$K_j$用以下式子表达：

$$[\varepsilon^0_j] = [\varepsilon^0_x, \varepsilon^0_y, \varepsilon^0_{xy}]^T \tag{4}$$

$$[K_j] = [k_x, k_y, k_{xy}]^T \tag{5}$$

其中：

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$$\varepsilon^0_x=\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2,\qquad \varepsilon^0_y=\frac{\partial v_0}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2$$

$$\varepsilon^0_{xy}=\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\tag{6}$$

$$k_x=\frac{\partial\psi_x}{\partial x},\qquad k_y=\frac{\partial\psi_y}{\partial y},\qquad k_{xy}=\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\tag{7}$$

（2）式中的$A_{ij}$，$B_{ij}$，$D_{ij}$分别为拉伸刚度、耦合刚度和弯曲刚度。

对于中面单位宽度的横向剪切力，定义：

$$[Q]=[Q_y,\ Q_x]^T=\int_{-h/2}^{h/2}[\tau_{yz},\ \tau_{xz}]^T dz\tag{8}$$

考虑到应力应变关系：

$$\begin{Bmatrix}Q_y\\Q_x\end{Bmatrix}=\begin{bmatrix}\overline{A}_{44}&\overline{A}_{45}\\\overline{A}_{45}&\overline{A}_{55}\end{bmatrix}\begin{Bmatrix}w_{,y}+\psi_y\\w_{,x}+\psi_x\end{Bmatrix}\tag{9}$$

其中

$$\overline{A}_{ij}=k_ik_jA_{ij},\qquad A_{ij}=\int_{-h/2}^{h/2}c_{ij}dz\quad(i,\ j=4,\ 5)\tag{10}$$

$k_4$，$k_5$称为剪切修正系数。

考虑到板弯曲时的平衡关系，计及惯性力，运动方程如下：

$$\begin{aligned}&N_{x,x}+N_{xy,y}=R_1u_{0,tt}+R_2\psi_{x,tt}\\&N_{xy,x}+N_{y,y}=R_1v_{0,tt}+R_2\psi_{y,tt}\\&Q_{x,x}+Q_{y,y}+q+N_xw_{,xx}+N_yw_{,yy}+2N_{xy}w_{,xy}=R_1w_{,tt}\\&M_{x,x}+M_{xy,y}-Q_x=R_2u_{0,tt}+R_3\psi_{x,tt}\\&M_{xy,x}+M_{y,y}-Q_y=R_2v_{0,tt}+R_3\psi_{y,tt}\end{aligned}\tag{11}$$

这里

$$(R_1,\ R_2,\ R_3)=\int_{-h/2}^{h/2}\rho\ (1,\ z,\ z^2)\ dz,\qquad \rho_i=\frac{\gamma h}{g}\tag{12}$$

把上述有关的项代入（11）式中得到一般层合板的控制方程：

$$\begin{aligned}&L_1^Au_0+L_2^Av_0+L_1^B\psi_x+L_2^B\psi_y-R_1u_{0,tt}-R_2\psi_{x,tt}=-w_{,x}L_1^Aw-w_{,y}L_2^Aw\\&L_2^Au_0+L_3^Av_0+L_2^B\psi_x+L_3^B\psi_y-R_1v_{0,tt}-R_2\psi_{y,tt}=-w_{,x}L_2^Aw-w_{,y}L_3^Aw\\&\overline{A}_{44}\ (\psi_{y,y}+w_{,yy})\ +\overline{A}_{45}\ (\psi_{x,y}+\psi_{y,x}+2w_{,xy})\ +\overline{A}_{55}\ (\psi_{x,x}+w_{,xx})\ +q-R_1w_{,tt}=\overline{q}\end{aligned}\tag{13}$$

$$\begin{aligned}&L_1^Bu_0+L_2^Bv_0+L_1^D\psi_x+L_2^D\psi_y-\overline{A}_{45}\ (\psi_y+w_{,y})\ -\overline{A}_{55}\ (\psi_x+w_{,x})\ -R_2u_{0,tt}-\\&\qquad R_3\psi_{x,tt}=-w_{,x}L_1^Bw-w_{,y}L_2^Bw\\&L_2^Bu_0+L_3v_0+L_2^D\psi_x+L_3^D\psi_y-\overline{A}_{44}\ (\psi_y+w_{,y})\ -\overline{A}_{45}\ (\psi_x+w_{,x})\ -R_2v_{0,tt}-\\&\qquad -R_3\psi_{y,tt}=-w_{,x}L_2^Bw-w_{,y}L_3^Bw\\&\overline{q}=-L_4^Aw\ (u_{0,x}+\tfrac{1}{2}w_{,x}{}^2)\ -L_5^Aw\ (v_{0,y}+\tfrac{1}{2}w_{,}{}^2{}_y)\ -L_6^Aw\ (u_{0,y}+v_{0,x}\\&\qquad +w_{,x}w_{,y})\ -L_4^Bw\psi_{x,x}-L_5^Bw\psi_{y,y}-L_6^Bw\ (\psi_{x,y}+\psi_{y,x})\end{aligned}$$

（13）式中的$L_i^A$，$L_i^B$，$L_i^D$均为偏微分算子：

$$
\begin{aligned}
L_1^A &= A_{11}(\ )_{,xx}+2A_{16}(\ )_{,xy}+A_{66}(\ )_{,yy} \\
L_2^A &= A_{16}(\ )_{,xx}+(A_{12}+A_{66})(\ )_{,xy}+A_{26}(\ )_{,yy} \\
L_3^A &= A_{66}(\ )_{,xx}+2A_{26}(\ )_{,xy}+A_{22}(\ )_{,yy} \\
L_4^A &= A_{11}(\ )_{,xx}+2A_{16}(\ )_{,xy}+A_{12}(\ )_{,yy} \\
L_5^A &= A_{12}(\ )_{,xx}+2A_{26}(\ )_{,xy}+A_{22}(\ )_{,yy} \\
L_6^A &= A_{16}(\ )_{,xx}+2A_{66}(\ )_{,xy}+A_{26}(\ )_{,yy} \\
L_1^B &= B_{11}(\ )_{,xx}+2B_{16}(\ )_{,xy}+B_{66}(\ )_{,yy} \\
L_2^B &= B_{16}(\ )_{,xx}+(B_{12}+B_{66})(\ )_{,xy}+B_{26}(\ )_{,yy} \\
L_3^B &= B_{66}(\ )_{,xx}+2B_{26}(\ )_{,xy}+B_{22}(\ )_{,yy} \\
L_4^B &= B_{11}(\ )_{,xx}+2B_{16}(\ )_{,xy}+B_{12}(\ )_{,yy} \\
L_5^B &= B_{12}(\ )_{,xx}+2B_{26}(\ )_{,xy}+B_{22}(\ )_{,yy} \\
L_6^B &= B_{16}(\ )_{,xx}+2B_{66}(\ )_{,xy}+B_{26}(\ )_{,yy} \\
L_1^D &= D_{11}(\ )_{,xx}+2D_{16}(\ )_{,xy}+D_{66}(\ )_{,yy} \\
L_2^D &= D_{16}(\ )_{,xx}+(D_{12}+D_{66})(\ )_{,xy}+D_{26},(\ )_{yy} \\
L_3^D &= D_{66}(\ )_{,xx}+2D_{26}(\ )_{,xy}+D_{22}(\ )_{,yy}
\end{aligned} \tag{14}
$$

从（13）式中可以清楚地看出，等式左边为线性项，等式右边为非线性项。如果仅考虑线性问题，则删掉右边各项；如果仅考虑静态变形则删掉与时间t有关的各项。

## 二　几种较为典型的边界条件

考虑一x轴方向长度为a，y轴方向宽度为b的矩形层合板，原点o位于板的中心。最常见的边界条件为四边简支，四边固支，自由边界以及介于简支固支之间的弹性转动约束边界条件。每一边有五个边界条件。

1.四边简支（SS—1）

$$
\begin{aligned}
x &= \pm a/2, & w &= M_x=\psi_y=u=v=0 \\
y &= \pm b/2, & w &= M_y=\psi_x=v=u=0
\end{aligned} \tag{15}
$$

2.四边简支（SS—2）

$$
\begin{aligned}
x &= \pm a/2, & w &= M_x=\psi_y=N_x=N_{xy}=0 \\
y &= \pm b/2, & w &= M_y=\psi_x=N_y=N_{xy}=0
\end{aligned} \tag{16}
$$

3.四边简支（SS—3）

$$
\begin{aligned}
x &= \pm a/2, & w &= M_x=\psi_y=v=N_x=0 \\
y &= \pm b/2, & w &= M_y=\psi_x=u=N_y=0
\end{aligned} \tag{17}
$$

4.四边固支（CC—1）

$$
\begin{aligned}
x &= \pm a/2, & w &= \psi_x=\psi_y=u=v=0 \\
y &= \pm b/2, & w &= \psi_y=\psi_x=v=u=0
\end{aligned} \tag{18}
$$

5.四边固支（CC—2）

$$
\begin{aligned}
x &= \pm a/2, & w &= \psi_x=\psi_y=N_x=N_{xy}=0 \\
y &= \pm b/2, & w &= \psi_y=\psi_x=N_y=N_{xy}=0
\end{aligned} \tag{19}
$$

6.自由边界

$$x=\pm a/2,\qquad N_y=N_{xy}=M_y=M_{xy}=Q_y=0$$
$$y=\pm b/2,\qquad N_x=N_{xy}=M_x=M_{xy}=Q_x=0 \tag{20}$$

7.弹性转动约束支承

$$x=\pm a/2,\qquad w=\psi_y=0,\ \ \psi_{x,x}=\pm k_1\psi_x,\ \ N_x=N_{xy}=0$$
$$y=\pm b/2,\qquad w=\psi_x=0,\ \ \psi_{y,y}=\pm k_2\psi_y,\ \ N_y=N_{xy}=0 \tag{21}$$

上式中$k_1$、$k_2$称为弹性转动系数。$k_i=\infty$对应于四边固支的边界条件（CC—2）；如果层合板是正交异性材料，$k_i=0$则对应于简支边的边界条件（SS—2）。因此，适当选择$k_1$，$k_2$的值就可以研究介于简支边和固支边之间的各种不同边界条件的非线性问题。

## 三 结 论

复合材料及其结构的材料参数和几何参数较各向同性材料要多。未知函数、控制方程、边界条件和初始条件也增加了，这给问题的求解带来了不少的困难。对于某种特定的复合材料结构，（13）式可以得到某些简化，再根据具体情况选择本文所列的边界条件或它们的组合，利用适当的方法求解。在求解非线性问题时，原则上要既简单又经济。

## 参 考 文 献

〔1〕 杨加明、孙锁泰，弹性转动约束正交异性矩形板的几何非线性分析，江苏工学院学报，3，1989

〔2〕 杨加明，两邻边铰支两邻边固支矩形板的几何非线性分析，华东交通大学学报，2，1989

〔3〕 E.Reissner, On the theory of bending of elastic plates, J. Math. Phys., 23, 1944

〔4〕 J. N. Reddy, A refined nonlinear theory of plates with transverse Shear deformation, Int. J. Solids Struct., 1984

# Geometrically Nonlinear Basic Equations of Composite Laminted Plates

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ABSTACR

On the basis of Reissner-Mindlin's Shear-deformable theory in this paper, the problems of the geometric nonlinearity are discussedfor composite materials. Governing equations of general laminated plates are obtained; and seven typical boundary conditions are finally presented.