动力学概率分区广义变分原理

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摘　　要

本文建立了动力学概率分区变分原理及概率分区广义变分原理，它们是建立动力学概率分区有限元和概率分区广义有限元法的理论基础。

关键词：概率变分原理，分区变分，广义变分。

一、引　　言

现代工程结构的两大特点是：一是具有随机性（结构的几何参数、材料的物理性质及受力情况都是不确定的），二是结构日趋大型化和复杂化，因此合理地建立既能反映结构的随机性和又能有效地减缩系统自由度的计算模型是十分必要的。近几年来发展起来的随机有限元法是分析结构随机性的一种较好的方法，而动力子结构法则是用于减缩系统自由度的常用技巧。将随机有限元法和动力子结构技术集以一体将是分析现代动力工程结构的一条有效途径。随机有限元法的理论基础是概率变分原理，动力子结构法的理论基础是分区变分原理，建立概率分区变分原理和概率分区广义变分原理是发展相应的有限元法的基础和前提。本文在文献〔2〕的基础上，采用将随机变量作级数展开的方法，建立了动力学概率分区变分原理和概率分区广义变分原理，利用它们可建立动力学随机有限元方法。

二、基　本　方　程

不失一般性，本文讨论将弹性体的体积分为两部分，即：$\Omega=\Omega_1+\Omega_2$，Ω_1的边界为$\Gamma_{\sigma_1}+\Gamma_{u_2}+\Gamma_0$，$\Omega_2$的边界为$\Gamma_{\sigma_2}+\Gamma_{u_2}+\Gamma_0$，其中$\Gamma_{\sigma_1}$，$\Gamma_{\sigma_2}$是给定外力的边界，$\Gamma_{u_1}$，$\Gamma_{u_2}$是给定位移的边界，$\Omega_1$和$\Omega_2$的交界面为$\Gamma_0$。设在$\Gamma_0$上从$\Omega_1$指向$\Omega_2$的法向单位矢量为$\underline{n}^1$，从$\Omega_2$指向$\Omega_1$的法向单位矢量为$\underline{n}^2$，则有

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$$\underline{n}^1=-\underline{n}^2 \tag{1}$$

在$\Omega_\alpha(\alpha=1,2)$中的位移、应变和应力分别用$\underline{u}^\alpha$，$\underline{\varepsilon}^\alpha$和$\underline{\sigma}^\alpha$表示，则动力学基本方程为

运动方程

$$\underline{E}(\partial)\underline{\sigma}^\alpha+\underline{F}^\alpha=\rho^\alpha\ddot{\underline{u}}^\alpha \qquad (\text{in } \Omega_\alpha) \tag{2}$$

几何方程

$$\underline{\varepsilon}^\alpha=\underline{E}(\partial)^T\underline{u}^\alpha \qquad (\text{in } \Omega_\alpha) \tag{3}$$

物理方程

$$\underline{\sigma}^\alpha=\underline{D}^\alpha\underline{\varepsilon}^\alpha \qquad (\text{in } \Omega_\alpha) \tag{4}$$

位移、应力边界条件

$$\underline{u}^\alpha=\overline{\underline{u}^\alpha} \qquad (\text{on } \Gamma_{u_\alpha}) \tag{5}$$

$$\underline{E}(\underline{n}^\alpha)\underline{\sigma}^\alpha=\overline{\underline{P}^\alpha} \qquad (\text{on } \Gamma_{\sigma_\alpha}) \tag{6}$$

时端位移条件

$$\underline{u}^\alpha(t_1)=\overline{\underline{u}^\alpha}(t_1),\qquad \underline{u}^\alpha(t_2)=\overline{\underline{u}^\alpha}(t_2) \tag{7}$$

交界面位移连续条件

$$\underline{u}^1=\underline{u}^2 \qquad (\text{on } \Gamma_0) \tag{8}$$

其中：$\alpha=1,2$

$\underline{\sigma}^\alpha=\{\sigma_x{}^\alpha,\ \sigma_y{}^\alpha,\ \sigma_z{}^\alpha,\ \tau_{yz}{}^\alpha,\ \tau_{zx}{}^\alpha,\ \tau_{xy}{}^\alpha\}^T$

$\underline{\varepsilon}^\alpha=\{\varepsilon_x{}^\alpha,\ \varepsilon_y{}^\alpha,\ \varepsilon_z{}^\alpha,\ \gamma_{yz}{}^\alpha,\ \gamma_{zx}{}^\alpha,\ \gamma_{xy}{}^\alpha\}^T$

$\underline{u}^\alpha=\{u^\alpha,\ v^\alpha,\ w^\alpha\}^T$

$\underline{F}^\alpha=\{F_x{}^\alpha,\ F_y{}^\alpha,\ F_z{}^\alpha\}^T$

$\overline{\underline{P}^\alpha}=\{\overline{P_x{}^\alpha},\ \overline{P_y{}^\alpha},\ \overline{P_z{}^\alpha}\}^T$

$\underline{E}(\partial)$为微分算子矩阵[7]，$\underline{E}(\partial)$为

$$\underline{E}(\partial)=\begin{pmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{pmatrix}$$

$\underline{E}(n)$为方向余弦矩阵，在$\underline{E}(n)$中，用n_i代换$\frac{\partial}{\partial x_i}$即得到$\underline{E}(n)$。

三、动力学概率分区变分方程

动力学分区变分原理的总势能泛函为〔2〕

$$\Pi^I=\sum_\alpha\int_{t_1}^{t_2}dt\left\{\frac{1}{2}\int_{\Omega_\alpha}\left(\rho^\alpha\underline{\dot{u}}^{\alpha T}\underline{\dot{u}}^\alpha-\underline{\varepsilon}^{\alpha T}\underline{D}^\alpha\underline{\varepsilon}^\alpha\right)d\Omega\right.$$

$$\left.+\int_{\Omega_\alpha}\underline{F}^{\alpha T}\underline{u}^\alpha d\Omega+\int_{\Gamma_{\sigma_\alpha}}\underline{\overline{P}}^\alpha\underline{u}^\alpha d\Gamma\right\} \tag{9}$$

在概率问题中，$\underline{u}^\alpha$，$\underline{\dot{u}}^\alpha$，$\underline{\varepsilon}^\alpha$，$\underline{D}^\alpha$，$\underline{\rho}^\alpha$，$\underline{F}^\alpha$及$\underline{\overline{P}}^\alpha$都是随机变量，因此总势能$\Pi^I$也是一个随机变量（本文不考虑几何形状的随机性），根据文献〔1〕〔4〕，可将这些随机变量在其均值处按Taylor级数展开，即

$$\left\{\begin{array}{ll}\underline{u}^\alpha=\sum\limits_{i=0}^{\infty}\underline{u_i}^\alpha\beta^i & \underline{\dot{u}}^\alpha=\sum\limits_{i=0}^{\infty}\underline{\dot{u}_i}^\alpha\beta^i\\ \underline{\varepsilon}^\alpha=\sum\limits_{i=0}^{\infty}\underline{\varepsilon_i}^\alpha\beta^i & D^\alpha=\sum\limits_{i=0}^{\infty}\underline{D_i}^\alpha\beta_i\\ \rho^\alpha=\sum\limits_{i=0}^{\infty}\rho_i^\alpha\beta^i & \underline{F}^\alpha=\sum\limits_{i=0}^{\infty}\underline{F_i}^\alpha\beta^i\\ \underline{\overline{P}}^\alpha=\sum\limits_{i=0}^{\infty}\underline{\overline{P_i}}^\alpha\beta^i & \Pi^I=\sum\limits_{i=0}^{\infty}\Pi_i{}^I\beta^i\end{array}\right. \tag{10}$$

式中，$\underline{u_i}^\alpha$，$\underline{\dot{u}_i}^\alpha$，$\underline{\varepsilon_i}^\alpha$，$\underline{D_i}^\alpha$，$\rho_i^\alpha$，$\underline{F_i}^\alpha$，$\underline{\overline{P_i}}^\alpha$及$\Pi_i{}^I$为其i阶展开系数，而零阶系数即为均值，$\beta^i$为展开系数。

将(10)式代入(9)中，经过整理及比较β^i相同的次数可得$\Pi_i{}^I$，利用变分原理可得

$$\delta\Pi_i{}^I=0\qquad(i=0,\ 1,\ 2,\ \cdots) \tag{11}$$

利用式(11)及边界条件可以建立动力学概率变分原理（推导方法与下节四中的推导基本相同）

$$\left\{\begin{array}{ll}\delta\Pi_{2k}{}^I=0 & (k=0,\ 1,\ 2,\ 3,\ \cdots)\\ \delta\Pi_{2k+1}{}^I\equiv0 & \end{array}\right. \tag{12}$$

式中：

$$\Pi_{2k}{}^I=\sum_\alpha\int_{t_1}^{t_2}\left\{\frac{1}{2}\int_{\Omega_\alpha}\left(\rho_0{}^\alpha\underline{\dot{u}_k}^{\alpha T}\underline{\dot{u}_k}^\alpha+2\rho_1{}^\alpha\underline{\dot{u}_k}^{\alpha T}\underline{\dot{\breve{u}}_{k-1}}^\alpha+2\rho_2{}^\alpha\underline{\dot{u}_k}^{\alpha T}\underline{\dot{\breve{u}}_{k-2}}^\alpha+\cdots+\right.\right.$$

$$2\rho_{k-1}{}^\alpha\underline{\dot{u}_k}^{\alpha T}\underline{\dot{\breve{u}}_1}+2\rho_k{}^\alpha\underline{\dot{u}_k}^{\alpha T}\underline{\dot{\breve{u}}_0}^\alpha-\underline{\varepsilon_k}^{\alpha T}\underline{D_0}^\alpha\underline{\varepsilon_k}^\alpha-2\underline{\varepsilon_k}^{\alpha T}\underline{D_1}^\alpha\underline{\breve{\varepsilon}_{k-1}}^\alpha-$$

$$\left.2\underline{\varepsilon_k}^{\alpha T}\underline{D_2}^\alpha\underline{\breve{\varepsilon}_{k-2}}^\alpha-\cdots-2\underline{\varepsilon_k}^{\alpha T}\underline{D_{k-1}}^\alpha\underline{\breve{\varepsilon}_1}^\alpha-2\underline{\varepsilon_k}^{\alpha T}\underline{D_k}^\alpha\underline{\breve{\varepsilon}_0}^\alpha\right)d\Omega+$$

$$\left.\int_{\Omega_\alpha}\underline{F_k}^{\alpha T}\underline{u_k}^\alpha d\Omega+\int_{\Gamma_{\sigma_\alpha}}\underline{\overline{P_k}}^{\alpha T}\underline{u_k}^\alpha d\Gamma\right\}dt \tag{13}$$

(13) 式是弹性动力学概率变分原理的泛函，其中$\breve{\underline{u}}_i^{\alpha}$及$\breve{\underline{\varepsilon}}_i^{\alpha}$为已求得的量。

弹性动力学概率分区变分原理可陈述如下：

对任意时刻 t，在边界Γ_{u_α}上满足已给的边界条件（5）和在区域Ω_α中满足几何关系（3）式，而在Γ_0上满足位移条件（8）式，那么在满足时端位移条件（7）式的一切可能运动中，实际的运动$\underline{u}_k^{\alpha}$，$\underline{\varepsilon}_k^{\alpha}$，$\underline{\sigma}_k^{\alpha}$，必使泛函$\Pi_{2k}{}^{I}$取驻值，而

$$\delta\Pi_{2k+1}{}^{I}=0$$

四、动力学概率分区广义变分原理

上节三中所导出的概率分区变分原理存在下列约束条件：应力应变关系（3）式，已知的位移边界条件（5）式，以及各区域间的位移连续条件（8）式和给定的时端位移条件（7）式。

下面我们讨论交界面位移连续条件（8）式的放松，通过引入拉格朗日乘子并加以识别的方法[6]，可得到相应的泛函为〔2〕，

$$\Pi^{I}=\sum_{\alpha}\int_{t_1}^{t_2}\Big\{\frac{1}{2}\int_{\Omega_\alpha}\Big(\rho^{\alpha}\dot{\underline{u}}^{\alpha T}\dot{\underline{u}}^{\alpha}-\underline{\varepsilon}^{\alpha T}\underline{D}^{\alpha}\underline{\varepsilon}^{\alpha}\Big)d\Omega+\int_{\Omega_\alpha}\underline{F}^{\alpha T}\underline{u}^{\alpha}d\Omega+\int_{\Gamma_{\sigma_\alpha}}\overline{\underline{P}^{\alpha}}\underline{u}^{\alpha}d\Gamma+$$

$$\frac{1}{2}\int_{\Gamma_0}[\underline{E}(\underline{n}^{\alpha})\underline{\sigma}^{\alpha}]^{T}(\underline{u}^{\alpha}-\underline{u}^{\alpha'})d\Gamma\Big\}dt \tag{14}$$

上式中，α'表示与α对立的编号。

在概率问题中，$\underline{u}^{\alpha}$，$\dot{\underline{u}}^{\alpha}$，$\underline{\varepsilon}^{\alpha}$，$\rho^{\alpha}$，$\underline{D}^{\alpha}$，$\underline{\sigma}^{\alpha}$，$\underline{F}^{\alpha}$及$\overline{\underline{P}}^{\alpha}$都是随机变量，随机变量可以在其均值处按Taylor级数展开

$$\begin{cases}\underline{u}^{\alpha}=\sum_{i=0}^{\infty}\underline{u}_i^{\alpha}\beta^{i} & \dot{\underline{u}}^{\alpha}=\sum_{i=0}^{\infty}\dot{\underline{u}}_i^{\alpha}\beta^{i} & \underline{\varepsilon}^{\alpha}=\sum_{i=0}^{\infty}\underline{\varepsilon}_i^{\alpha}\beta^{i}\\ \rho^{\alpha}=\sum_{i=0}^{\infty}\rho_i^{\alpha}\beta^{i} & \underline{D}^{\alpha}=\sum_{i=0}^{\infty}\underline{D}_i^{\alpha}\beta^{i} & \underline{\sigma}^{\alpha}=\sum_{i=0}^{\infty}\underline{\sigma}_i^{\alpha}\beta^{i}\\ \underline{F}^{\alpha}=\sum_{i=0}^{\infty}\underline{F}_i^{\alpha}\beta^{i} & \overline{\underline{P}}^{\alpha}=\sum_{i=0}^{\infty}\overline{\underline{P}}_i^{\alpha}\beta^{i} & \Pi^{I}=\sum_{i=0}^{\infty}\Pi_i^{I}\beta^{i}\end{cases} \tag{15}$$

式中，$\underline{u}_i^{\alpha}$，$\dot{\underline{u}}_i^{\alpha}$，$\underline{\varepsilon}_i^{\alpha}$，$\rho_i^{\alpha}$，$\underline{D}_i^{\alpha}$，$\underline{\sigma}_i^{\alpha}$，$\underline{F}_i^{\alpha}$，$\overline{\underline{P}}_i^{\alpha}$及$\Pi_i^{I}$为$\beta^{i}$的i阶展开系数，$\beta^{i}$为展开微量，将(15)式代入(14)中，经过整理及比较β^{i}的相同幂次可得Π_i^{I}，

$$\Pi_0{}^{I}=\sum_{\alpha}\int_{t_1}^{t_2}dt\Big\{\frac{1}{2}\int_{\Omega_\alpha}\Big(\rho_0{}^{\alpha}\dot{\underline{u}}_0{}^{\alpha T}\dot{\underline{u}}_0{}^{\alpha}-\underline{\varepsilon}_0{}^{\alpha T}\underline{D}_0{}^{\alpha}\underline{\varepsilon}_0{}^{\alpha}\Big)d\Omega+\int_{\Omega_\alpha}\underline{F}_0{}^{\alpha T}\underline{u}_0{}^{\alpha}d\Omega+$$

$$\int_{\Gamma_{\sigma_\alpha}}\overline{\underline{P}}_0{}^{\alpha T}\underline{u}_0{}^{\alpha}d\Gamma+\frac{1}{2}\int_{\Gamma_0}\Big[\underline{E}(\underline{n}^{\alpha})\underline{\sigma}_0{}^{\alpha}\Big]^{T}(\underline{u}_0{}^{\alpha}-\underline{u}_0{}^{\alpha'})d\Gamma\Big\} \tag{16}$$

$$\Pi_1{}^{I}=\sum_{\alpha}\int_{t_1}^{t_2}dt\Big\{\frac{1}{2}\int_{\Omega_\alpha}\Big(\rho_1{}^{\alpha}\dot{\underline{u}}_0{}^{\alpha T}\dot{\underline{u}}_0{}^{\alpha}+2\rho_0{}^{\alpha}\dot{\underline{u}}_0{}^{\alpha T}\dot{\underline{u}}_1{}^{\alpha}-\underline{\varepsilon}_0{}^{\alpha T}\underline{D}_1{}^{\alpha}\underline{\varepsilon}_0{}^{\alpha}-$$

$$2\underline{\varepsilon}_0^{\alpha T}\underline{D}_0^{\alpha}\underline{\varepsilon}_1^{\alpha}\Big)d\Omega+\int_{\Omega_\alpha}\Big(\underline{F}_0^{\alpha T}\underline{u}_1^{\alpha}+\underline{F}_1^{\alpha T}\underline{u}_0^{\alpha}\Big)d\Omega+$$

$$\int_{\Gamma_{\sigma_\alpha}}\Big(\underline{\overline{P}}_0^{\alpha T}\underline{u}_1^{\alpha}+\underline{\overline{P}}_1^{\alpha T}\underline{u}_0^{\alpha}\Big)d\Gamma+\frac{1}{2}\int_{\Gamma_0}\Big[\underline{E}(\underline{n}^{\alpha})\underline{\sigma}_0^{\alpha}\Big]^T\Big(\underline{u}_1^{\alpha}-$$

$$\underline{u}_1^{\alpha'}\Big)d\Gamma+\frac{1}{2}\int_{\Gamma_0}\Big[\underline{E}(\underline{n}^{\alpha})\underline{\sigma}_1^{\alpha}\Big]^T\Big(\underline{u}^{0\alpha}-\underline{u}_0^{\alpha'}\Big)d\Gamma\Big\} \qquad (17)$$

$$\Pi_2^{I}=\sum_{\alpha}\int_{t_1}^{t_2}dt\Big\{\frac{1}{2}\int_{\Omega_\alpha}\Big(\rho_2^{\alpha}\dot{\underline{u}}_0^{\alpha T}\dot{\underline{u}}_0+2\rho_0^{\alpha}\dot{\underline{u}}_0^{\alpha T}\dot{\underline{u}}_2^{\alpha}+2\rho_1^{\alpha}\dot{\underline{u}}_0^{\alpha T}\dot{\underline{u}}_1^{\alpha}+\rho_0^{\alpha}\dot{\underline{u}}_1^{\alpha T}\dot{\underline{u}}_1^{\alpha}-$$

$$\underline{\varepsilon}_0^{\alpha T}\underline{D}_2^{\alpha}\underline{\varepsilon}_0^{\alpha}-2\underline{\varepsilon}_0^{\alpha T}\underline{D}_0^{\alpha}\underline{\varepsilon}_2^{\alpha}-2\underline{\varepsilon}_0^{\alpha T}\underline{D}_1^{\alpha}\underline{\varepsilon}_1^{\alpha}-\underline{\varepsilon}_1^{\alpha T}\underline{D}_0^{\alpha}\underline{\varepsilon}_1^{\alpha}\Big)d\Omega+$$

$$\int_{\Omega_\alpha}\Big(\underline{F}_0^{\alpha T}\underline{u}_2^{\alpha}+\underline{F}_1^{\alpha T}\underline{u}_1^{\alpha}+\underline{F}_2^{\alpha T}\underline{u}_0^{\alpha}\Big)d\Omega+\int_{\Gamma_{\sigma_\alpha}}\Big(\underline{\overline{P}}_0^{\alpha T}\underline{u}_2^{\alpha}+\underline{\overline{P}}_1^{\alpha T}\underline{u}_1^{\alpha}+$$

$$\underline{\overline{P}}_2^{\alpha T}\underline{u}_0^{T}\Big)d\Gamma+\frac{1}{2}\int_{\Gamma_0}\Big[\underline{E}(\underline{n}^{\alpha})\ \underline{\sigma}_0^{\alpha}\Big]^T\Big(\underline{u}_2^{\alpha}-\underline{u}_2^{\alpha'}\Big)d\Gamma+$$

$$\frac{1}{2}\int_{\Gamma_0}\Big[\underline{E}(\underline{n}^{\alpha})\ \underline{\sigma}_1^{\alpha}\Big]^T\Big(\underline{u}_1^{\alpha}-\underline{u}_1^{\alpha'}\Big)d\Gamma+\frac{1}{2}\int_{\Gamma_0}\Big[\underline{E}(\underline{n}^{\alpha})\ \underline{\sigma}_2^{\alpha}\Big]^T\Big(\underline{u}_0^{\alpha}-$$

$$\underline{u}_0^{\alpha'}\Big)d\Gamma\Big\} \qquad (18)$$

$$\Pi_3^{I}=\sum_{\alpha}\int_{t_1}^{t_2}dt\Big\{\frac{1}{2}\int_{\Omega_\alpha}\Big(\rho_3^{\alpha}\dot{\underline{u}}_0^{\alpha T}\dot{\underline{u}}_0^{\alpha}+2\rho_0^{\alpha}\dot{\underline{u}}_0^{\alpha T}\dot{\underline{u}}_3^{\alpha}+2\rho_2^{\alpha}\dot{\underline{u}}_0^{\alpha T}\dot{\underline{u}}_3^{\alpha}+2\rho_0^{\alpha}\dot{\underline{u}}_1^{\alpha T}\dot{\underline{u}}_2^{\alpha}+$$

$$2\rho_1^{\alpha}\dot{\underline{u}}_0^{\alpha T}\dot{\underline{u}}_2^{\alpha}+\rho_1^{\alpha}\dot{\underline{u}}_1^{\alpha T}\dot{\underline{u}}_1^{\alpha}-\underline{\varepsilon}_0^{\alpha T}\underline{D}_3^{\alpha}\underline{\varepsilon}_0^{\alpha}-2\underline{\varepsilon}_0^{\alpha T}\underline{D}_0^{\alpha}\underline{\varepsilon}_3^{\alpha}-2\underline{\varepsilon}_0^{\alpha T}\underline{D}_2^{\alpha}\underline{\varepsilon}_1^{\alpha}-$$

$$2\underline{\varepsilon}_1^{\alpha T}\underline{D}_0^{\alpha}\underline{\varepsilon}_2^{\alpha}-2\underline{\varepsilon}_0^{\alpha T}\underline{D}_1^{\alpha}\underline{\varepsilon}_2^{\alpha}-\underline{\varepsilon}_1^{\alpha T}\underline{D}_1^{\alpha}\underline{\varepsilon}_1^{\alpha}\Big)d\Omega+\int_{\Omega_\alpha}\Big(\underline{F}_0^{\alpha T}\underline{u}_3^{\alpha}+\underline{F}_1^{\alpha T}\underline{u}_2^{\alpha}+$$

$$\underline{F}_2^{\alpha T}\underline{u}_1^{\alpha}+\underline{F}_3^{\alpha T}\underline{u}_0^{\alpha}\Big)d\Omega+\int_{\Gamma_{\sigma_\alpha}}\Big(\underline{\overline{P}}_0^{\alpha T}\underline{u}_3^{\alpha}+\underline{\overline{P}}_1^{\alpha T}\underline{u}_2^{\alpha}+\underline{\overline{P}}_2^{\alpha T}\underline{u}_1^{\alpha}+$$

$$\underline{\overline{P}}_3^{\alpha T}\underline{u}_0^{\alpha}\Big)d\Gamma+\frac{1}{2}\int_{\Gamma_0}\Big[\underline{E}(\underline{n}^{\alpha})\underline{\sigma}_0^{\alpha}\Big]^T\Big(\underline{u}_3^{\alpha}-\underline{u}_3^{\alpha'}\Big)d\Gamma+$$

$$\frac{1}{2}\int_{\Gamma_0}\Big[\underline{E}(\underline{n}^{\alpha})\underline{\sigma}_1^{\alpha}\Big]^T\Big(\underline{u}_2^{\alpha}-\underline{u}_2^{\alpha'}\Big)d\Gamma+\frac{1}{2}\int_{\Gamma_0}\Big[\underline{E}(\underline{n}^{\alpha})\underline{\sigma}_2^{\alpha}\Big]^T\Big(\underline{u}_1^{\alpha}-$$

$$\underline{u}_1^{\alpha'}\Big)d\Gamma+\frac{1}{2}\int_{\Gamma_0}\Big[\underline{E}(\underline{n}^{\alpha})\underline{\sigma}_3^{\alpha}\Big]^T\Big(\underline{u}_0^{\alpha}-\underline{u}_0^{\alpha'}\Big)d\Gamma\Big\} \qquad (19)$$

$$\Pi_4^{I}=\sum_{\alpha}\int_{t_1}^{t_2}dt\Big\{\frac{1}{2}\int_{\Omega_\alpha}\Big(\rho_4^{\alpha}\dot{\underline{u}}_0^{\alpha T}\dot{\underline{u}}_0^{\alpha}+2\rho_0^{\alpha}\dot{\underline{u}}_0^{\alpha T}\dot{\underline{u}}_4^{\alpha}+2\rho_3^{\alpha}\dot{\underline{u}}_0^{\alpha T}\dot{\underline{u}}_1^{\alpha}+2\rho_1^{\alpha}\dot{\underline{u}}_0^{\alpha T}\dot{\underline{u}}_3^{\alpha}+$$

$$2\rho_0^{\alpha}\dot{\underline{u}}_1^{\alpha T}\dot{\underline{u}}_3^{\alpha}+2\rho_0^{\alpha}\dot{\underline{u}}_0^{\alpha T}\dot{\underline{u}}_2^{\alpha}+2\rho_1^{\alpha}\dot{\underline{u}}_1^{\alpha T}\dot{\underline{u}}_2^{\alpha}+\rho_2\dot{\underline{u}}_1^{\alpha T}\dot{\underline{u}}_1^{\alpha}+\rho_0^{\alpha}\dot{\underline{u}}_1^{\alpha T}\dot{\underline{u}}_2^{\alpha}-$$

$$\underline{\varepsilon}_0^{\alpha T}\underline{D}_4^{\alpha}\underline{\varepsilon}_0^{\alpha}-2\underline{\varepsilon}_0^{\alpha T}\underline{D}_0^{\alpha}\underline{\varepsilon}_4^{\alpha}-2\underline{\varepsilon}_0^{\alpha T}\underline{D}_3^{\alpha}\underline{\varepsilon}_1^{\alpha}-2\underline{\varepsilon}_0^{\alpha T}\underline{D}_1^{\alpha}\underline{\varepsilon}_3^{\alpha}-2\underline{\varepsilon}_1^{\alpha T}\underline{D}_0^{\alpha}\underline{\varepsilon}_3^{\alpha}-$$

$$2\underline{\varepsilon_0}^{\alpha T}\underline{D_2}^{\alpha}\underline{\varepsilon_2}^{\alpha}-2\underline{\varepsilon_1}^{\alpha T}\underline{D_1}^{\alpha}\underline{\varepsilon_2}^{\alpha}-\underline{\varepsilon_1}^{\alpha T}\underline{D_2}^{\alpha}\underline{\varepsilon_1}^{\alpha}-\underline{\varepsilon_2}^{\alpha T}\underline{D_0}^{\alpha}\underline{\varepsilon_2}^{\alpha}\Big)d\Omega+$$

$$\int_{\Omega_\alpha}\Big(\underline{F_0}^{\alpha T}\underline{u_4}^{\alpha}+\underline{F_1}^{\alpha T}\underline{u_3}^{\alpha}+\underline{F_2}^{\alpha T}\underline{u_2}^{\alpha}+\underline{F_3}^{\alpha T}\underline{u_1}^{\alpha}+\underline{F_4}^{\alpha T}\underline{u_0}^{\alpha}\Big)d\Omega+$$

$$\int_{\Gamma_{\sigma_\alpha}}\Big(\underline{\overline{P_0}}^{\alpha T}\underline{u_4}^{\alpha}+\underline{\overline{P_1}}^{\alpha T}\underline{u_3}^{\alpha}+\underline{\overline{P_2}}^{\alpha T}\underline{u_2}^{\alpha}+\underline{\overline{P_3}}^{\alpha T}\underline{u_1}^{\alpha}+\underline{\overline{P_4}}^{\alpha T}\underline{u_0}^{\alpha}\Big)d\Gamma+$$

$$\frac{1}{2}\int_{\Gamma_0}\Big[\underline{E}(\underline{n}^{\alpha})\underline{\sigma_0}^{\alpha}\Big]^T\Big(\underline{u_4}^{\alpha}-\underline{u_4}^{\alpha'}\Big)d\Gamma+\frac{1}{2}\int_{\Gamma_0}\Big[\underline{E}(\underline{n}^{\alpha})\underline{\sigma_1}^{\alpha}\Big]^T\Big(\underline{u_3}^{\alpha}-$$

$$\underline{u_3}^{\alpha'}\Big)d\Gamma+\frac{1}{2}\int_{\Gamma_0}\Big[\underline{E}(\underline{n}^{\alpha})\underline{\sigma_2}^{\alpha}\Big]^T\Big(\underline{u_2}^{\alpha}-\underline{u_2}^{\alpha'}\Big)d\Gamma+\frac{1}{2}\int_{\Gamma_0}\Big[\underline{E}(\underline{n}^{\alpha})\underline{\sigma_3}^{\alpha}\Big]^T\cdot$$

$$\Big(\underline{u_1}^{\alpha}-\underline{u_1}^{\alpha'}\Big)d\Gamma+\frac{1}{2}\int_{\Gamma_0}\Big[\underline{E}(\underline{n}^{\alpha})\underline{\sigma_4}^{\alpha}\Big]^T\Big(\underline{u_0}^{\alpha}-\underline{u_0}^{\alpha'}\Big)d\Gamma\Big\} \tag{20}$$

利用变分原理可得

$$\delta\Pi_i{}^I=0 \qquad (i=0,\ 1,\ 2,\ 3,\ \cdots) \tag{21}$$

由（16）式，得到

$$\delta\Pi_0{}^I=\sum_\alpha\int_{t_1}^{t_2}dt\Big\{\int_{\Omega_\alpha}\Big(\rho_0{}^{\alpha}\dot{\underline{u}}_0{}^{\alpha T}\delta\dot{\underline{u}}_0{}^{\alpha}-\underline{\varepsilon_0}^{\alpha T}\underline{D_0}^{\alpha}\delta\underline{\varepsilon_0}^{\alpha}\Big)d\Omega+\int_{\Omega_\alpha}\underline{F_0}^{\alpha T}\delta\underline{u_0}^{\alpha}d\Omega+$$

$$\int_{\Gamma_{\sigma_\alpha}}\underline{\overline{P_0}}^{\alpha T}\delta\underline{u_0}^{\alpha}d\Gamma+\frac{1}{2}\int_{\Gamma_0}\Big[\underline{E}(\underline{n}^{\alpha})\delta\underline{\sigma_0}^{\alpha}\Big]^T\Big(\underline{u_0}^{\alpha}-\underline{u_0}^{\alpha'}\Big)d\Gamma+$$

$$\frac{1}{2}\int_{\Gamma_0}\Big[\underline{E}(\underline{n}^{\alpha})\underline{\sigma_0}^{\alpha}\Big]^T\Big(\delta\underline{u_0}^{\alpha}-\delta\underline{u_0}^{\alpha'}\Big)d\Gamma\Big\} \tag{22}$$

因为$\int_{t_1}^{t_2}\int_{\Omega_\alpha}\rho_0{}^{\alpha}\dot{\underline{u}}_0{}^{\alpha T}\delta\underline{u_0}^{\alpha}d\Omega dt=\int_{t_1}^{t_2}\int_{\Omega_\alpha}\Big[\rho_0{}^{\alpha}\frac{\partial}{\partial t}\Big(\dot{\underline{u}}_0{}^{\alpha T}\delta\underline{u}^{0\alpha}\Big)-\rho_0{}^{\alpha}\ddot{\underline{u}}_0{}^{\alpha T}\delta\underline{u_0}^{\alpha}\Big]d\Omega dt$

$$=\int_{\Omega_\alpha}\Big[\rho_0{}^{\alpha}\dot{\underline{u}}_0{}^{\alpha T}\delta\underline{u_0}^{\alpha}\Big|_{t_1}^{t_2}-\int_{t_1}^{t_2}\rho_0{}^{\alpha}\ddot{\underline{u}}_0{}^{\alpha T}\delta\underline{u_0}^{\alpha}dt\Big]d\Omega$$

$$=-\int_{t_1}^{t_2}\int_{\Omega_\alpha}\rho_0{}^{\alpha}\ddot{\underline{u}}_0{}^{\alpha T}\delta\underline{u_0}^{\alpha}d\Omega dt$$

及$-\int_{t_1}^{t_2}\int_{\Omega_\alpha}\underline{\varepsilon_0}^{\alpha T}\underline{D_0}^{\alpha}\delta\underline{\varepsilon_0}^{\alpha}d\Omega dt=-\int_{t_1}^{t_2}\int_{\Gamma_{\sigma_\alpha}+\Gamma^0}\Big[\underline{E}(\underline{n}^{\alpha})\underline{\sigma_0}^{\alpha}\Big]^T\delta\underline{u}^{0\alpha}d\Gamma d\tau+$

$$\int_{t_1}^{t_2}\int_{\Omega_\alpha}\Big[\underline{E}(\partial)\underline{\sigma_0}^{\alpha}\Big]^T\delta\underline{u_0}d\Omega dt$$

又因为$\sum_\alpha\int_{\Gamma_0}\Big[\underline{E}(\underline{n}^{\alpha})\underline{\sigma_0}^{\alpha}\Big]^T\delta u_0{}^{\alpha'}d\Gamma=\sum_\alpha\int_{\Gamma_0}\Big[\underline{E}(\underline{n}^{\alpha'})\underline{\sigma_0}^{\alpha'}\Big]^T\delta\underline{u_0}^{\alpha}d\Gamma$

将以上各式代入（22）中，经整理得

$$\delta\Pi_0{}^I=\sum_\alpha\int_{t_1}^{t2}dt\Big\{\int_{\Omega_\alpha}\Big(-\rho_0{}^{\alpha}\ddot{\underline{u}}_0{}^{\alpha}+\Big[\underline{E}(\partial)\underline{\sigma_0}^{\alpha}\Big]+\underline{F_0}^{\alpha}\Big)^T\delta\underline{u_0}^{\alpha}d\Omega dt-$$

$$\int_{\Gamma_{\sigma_\alpha}}\Big(\Big[\underline{E}(\underline{n}^{\alpha})\underline{\sigma_0}^{\alpha}\Big]-\underline{\overline{P_0}}^{\alpha}\Big)^T\delta\underline{u_0}^{\alpha}d\Gamma dt+\frac{1}{2}\int_{\Gamma_0}\Big[\underline{E}(\underline{n}^{\alpha})\delta\underline{\sigma_0}^{\alpha}\Big]^T\cdot$$

$$\left(\underline{u}_0^{\alpha}-\underline{u}_0^{\alpha'}\right)d\Gamma-\frac{1}{2}\int_{\Gamma_0}\left(\left[\underline{E}(\underline{n}^{\alpha})\underline{\sigma}_0^{\alpha}\right]+\left[\underline{E}(\underline{n}^{\alpha'})\underline{\sigma}_0^{\alpha'}\right]\right)^T\delta\underline{u}_0^{\alpha}d\Gamma\}$$ (23)

由（21）式，有

$$\delta\Pi_0^{\mathrm{I}}=0$$ (24)

可见，只要所选择的位移函数满足位移边界条件、时端位移条件，当总势能泛函取极值时，则动力学运动方程，内力边界条件及区域交接面上的位移、内力连续条件自然得到满足。

由（24）式可求出$\overset{\frown}{\underline{u}_0}{}^{\alpha}$，$\overset{\frown}{\underline{\sigma}_0}{}^{\alpha}$，将$\overset{\frown}{\underline{u}_0}{}^{\alpha}$，$\overset{\frown}{\underline{\sigma}_0}{}^{\alpha}$代入（17）式中的$\Pi_1^{\mathrm{I}}$，则有

$$\delta\Pi_1^{\mathrm{I}}=\sum_{\alpha}\int_{t_1}^{t_2}dt\left\{\int_{\Omega_\alpha}\left(\rho_0^{\alpha}\dot{\tilde{\underline{u}}}_0^{\alpha T}\delta\dot{\underline{u}}_1^{\alpha}-\tilde{\underline{\varepsilon}}_0^{\alpha T}\underline{D}_0^{\alpha}\delta\underline{\varepsilon}_1^{\alpha}\right)d\Omega+\int_{\Omega_\alpha}\underline{F}_0^{\alpha T}\delta\underline{u}_1^{\alpha}d\Omega+\right.$$

$$\int_{\Gamma_{\sigma_\alpha}}\overline{\underline{P}}_0^{\alpha T}\delta\underline{u}_1^{\alpha}d\Gamma+\frac{1}{2}\int_{\Gamma_0}\left[\underline{E}(\underline{n}^{\alpha})\tilde{\underline{\sigma}}_0^{\alpha}\right]^T\left(\delta\underline{u}_1^{\alpha}-\delta\underline{u}_1^{\alpha'}\right)d\Gamma+$$

$$\left.\frac{1}{2}\int_{\Gamma_0}\left[\underline{E}(\underline{n}^{\alpha})\delta\underline{\sigma}_1^{\alpha}\right]^T\left(\tilde{\underline{u}}_0^{\alpha}-\tilde{\underline{u}}_0^{\alpha'}\right)d\Gamma\right\}$$ (25)

仿（22）式的化简方法，容易证明

$$\delta\Pi_1^{\mathrm{I}}\equiv0$$ (26)

将$\overset{\frown}{\underline{u}_0}{}^{\alpha}$，$\overset{\frown}{\underline{\sigma}_0}{}^{\alpha}$代入（18）式，变分后得

$$\delta\Pi_2^{\mathrm{I}}=\sum_{\alpha}\int_{t_1}^{t_2}dt\left\{\int_{\Omega_\alpha}\left(\rho_0\dot{\overset{\frown}{\underline{u}}}_0^{\alpha T}\delta\dot{\underline{u}}_2^{\alpha}+\rho_1^{\alpha}\dot{\overset{\frown}{\underline{u}}}_1\delta\dot{\underline{u}}_1^{\alpha}+\rho_0^{\alpha}\dot{\underline{u}}_0^{\alpha T}\delta\dot{\underline{u}}_1^{\alpha}-\overset{\frown}{\underline{\varepsilon}}_0^{\alpha T}\underline{D}_0^{\alpha}\delta\underline{\varepsilon}_2^{\alpha}-\right.\right.$$

$$\left.\overset{\frown}{\underline{\varepsilon}}_0^{\alpha T}\underline{D}_1^{\alpha}\delta\underline{\varepsilon}_1^{\alpha}-\underline{\varepsilon}_1^{\alpha T}\delta\underline{\varepsilon}_1^{\alpha}\right)d\Omega+\int_{\Omega_\alpha}\left(\underline{F}_0^{\alpha T}D_0^{\alpha}\delta\underline{u}_2^{\alpha}+\underline{F}_1^{\alpha T}\delta\underline{u}_1^{\alpha}\right)d\Omega+$$

$$\int_{\Gamma_{\sigma_\alpha}}\left(\overline{\underline{P}}_0^{\alpha T}\delta\underline{u}_2^{\alpha}+\overline{\underline{P}}_1^{\alpha T}\delta\underline{u}_1^{\alpha}\right)d\Gamma+\frac{1}{2}\int_{\Gamma_0}\left[\underline{E}(\underline{n}^{\alpha})\overset{\frown}{\underline{\sigma}}_0^{\alpha}\right]^T\left(\delta\underline{u}_2^{\alpha}-\delta\underline{u}_2^{\alpha'}\right)d\Gamma+$$

$$\frac{1}{2}\int_{\Gamma_0}\left[\underline{E}(\underline{n}^{\alpha})\delta\underline{\sigma}_1^{\alpha}\right]^T\left(\underline{u}_1^{\alpha}-\underline{u}_1^{\alpha'}\right)d\Gamma+\frac{1}{2}\int_{\Gamma_0}\left[\underline{E}(\underline{n}^{\alpha})\underline{\sigma}_1^{\alpha}\right]^T\left(\delta\underline{u}_1^{\alpha}-\right.$$

$$\left.\left.\delta\underline{u}_1^{\alpha'}\right)d\Gamma+\frac{1}{2}\int_{\Gamma_0}\left[\underline{E}(\underline{n}^{\alpha})\delta\underline{\sigma}_2^{\alpha}\right]^T\left(\overset{\frown}{\underline{u}}_0^{\alpha}-\overset{\frown}{\underline{u}}_0^{\alpha'}\right)d\Gamma\right\}$$ (27)

同理，仿（22）式的化简方法，上式可写成

$$\delta\Pi_2^{\mathrm{I}}=\sum_{\alpha}\int_{t_1}^{t_2}dt\left\{\int_{\Omega_\alpha}\left(\rho_1^{\alpha}\dot{\overset{\frown}{\underline{u}}}_0^{\alpha T}\delta\dot{\underline{u}}_1^{\alpha}+\rho_0^{\alpha}\dot{\underline{u}}_1^{\alpha T}\delta\dot{\underline{u}}_1^{\alpha}-\overset{\frown}{\underline{\varepsilon}}_0^{\alpha T}\underline{D}_1^{\alpha}\delta\underline{\varepsilon}_1^{\alpha}-\right.\right.$$

$$\left.\underline{\varepsilon}_1^{\alpha T}\underline{D}_0^{\alpha}\delta\underline{\varepsilon}_1^{\alpha}\right)d\Omega+\int_{\Omega_\alpha}\underline{F}_1^{\alpha T}\delta\underline{u}_1^{\alpha}d\Omega+\int_{\Gamma_{\sigma_\alpha}}\overline{\underline{P}}_1^{\alpha T}\delta\underline{u}_1^{\alpha}d\Gamma+$$

$$\frac{1}{2}\int_{\Gamma_0}\left[\underline{E}(\underline{n}^{\alpha})\delta\underline{\sigma}_1^{\alpha}\right]^T\left(\underline{u}_1^{\alpha}-\underline{u}_1^{\alpha'}\right)d\Gamma+\frac{1}{2}\int_{\Gamma_0}\left[\underline{E}(\underline{n}^{\alpha})\underline{\sigma}_1^{\alpha}\right]^T\left(\delta\underline{u}_1^{\alpha}-\right.$$

$$\delta\underline{u}_1^{\alpha\prime}\Big)d\Gamma\Big\} \tag{28}$$

由$\delta\Pi_2^{I}=0$可求得$\overset{\frown}{\underline{u}}{}_1^{\alpha}$，$\overset{\frown}{\underline{\sigma}}{}_1^{\alpha}$，其中泛函$\Pi_2^{I}$为

$$\Pi_2^{I}=\sum_{\alpha}\int_{t_1}^{t_2}dt\Big\{\frac{1}{2}\int_{\Omega_\alpha}\Big(\rho_0^\alpha\dot{\underline{u}}_1^{\alpha T}\dot{\underline{u}}_1^{\alpha}+2\rho_1^\alpha\dot{\underline{u}}_1^{\alpha T}\dot{\overset{\frown}{\underline{u}}}{}_0^{\alpha}-\underline{\varepsilon}_1^{\alpha T}\underline{D}_0^{\alpha}\underline{\varepsilon}_1^{\alpha}-2\underline{\varepsilon}_1^{\alpha T}\underline{D}_1^{\alpha}\overset{\frown}{\underline{\varepsilon}}{}_0^{\alpha}\Big)d\Omega+\int_{\Omega_\alpha}\underline{F}_1^{\alpha T}\underline{u}_1^{\alpha}d\Omega+\int_{\Gamma_{\sigma_\alpha}}\overline{\underline{P}}_1^{\alpha T}\underline{u}_1^{\alpha}d\Gamma+\frac{1}{2}\int_{\Gamma_0}\Big[\underline{E}(\underline{n}^\alpha)\underline{\sigma}_1^{\alpha}\Big]^T\Big(\underline{u}_1^{\alpha}-\underline{u}_1^{\alpha\prime}\Big)d\Gamma\Big\} \tag{29}$$

将$\overset{\frown}{\underline{u}}{}_0^{\alpha}$，$\overset{\frown}{\underline{\sigma}}{}_0^{\alpha}$，$\overset{\frown}{\underline{u}}{}_1^{\alpha}$，$\overset{\frown}{\underline{\sigma}}{}_1^{\alpha}$代入（19）式中的$\Pi_3^{I}$，得

$$\delta\Pi_3^{I}=\sum_{\alpha}\int_{t_1}^{t_2}dt\Big\{\int_{\Omega_\alpha}\Big(\rho_0^\alpha\dot{\overset{\frown}{\underline{u}}}{}_0^{\alpha T}\delta\dot{\underline{u}}_2^{\alpha}+\rho_0^\alpha\dot{\overset{\frown}{\underline{u}}}{}_1^{\alpha T}\delta\dot{\underline{u}}_2+\rho_1^\alpha\dot{\overset{\frown}{\underline{u}}}{}_0^{\alpha T}\delta\dot{\underline{u}}_2^{\alpha}-\overset{\frown}{\underline{\varepsilon}}{}_0^{\alpha T}\underline{D}_0^{\alpha}\delta\underline{\varepsilon}_3^{\alpha}-\overset{\frown}{\underline{\varepsilon}}{}_1^{\alpha T}\underline{D}_0^{\alpha}\delta\underline{\varepsilon}_2^{\alpha}-\overset{\frown}{\underline{\varepsilon}}{}_0^{\alpha T}\underline{D}_1^{\alpha}\delta\underline{\varepsilon}_2^{\alpha}\Big)d\Omega+\int_{\Omega_\alpha}\Big(\underline{F}_0^{\alpha T}\delta\underline{u}_3^{\alpha}+\underline{F}_0^{\alpha T}\delta\underline{u}_2^{\alpha}\Big)d\Omega+\int_{\Gamma_{\sigma_\alpha}}\Big(\overline{\underline{P}}_0^{\alpha T}\delta\underline{u}_3^{\alpha}+\overline{\underline{P}}_0^{\alpha T}\delta\underline{u}_2^{\alpha}\Big)d\Gamma+\frac{1}{2}\int_{\Gamma_0}\Big[\underline{E}(\underline{n}^\alpha)\overset{\frown}{\underline{\sigma}}{}_0^{\alpha}\Big]^T\Big(\delta\underline{u}_3^{\alpha}-\delta\underline{u}_3^{\alpha\prime}\Big)d\Gamma+\frac{1}{2}\int_{\Gamma_0}\Big[\underline{E}(\underline{n}^\alpha)\overset{\frown}{\underline{\sigma}}{}_1^{\alpha}\Big]^T\Big(\delta\underline{u}_2^{\alpha}-\delta\underline{u}_2^{\alpha\prime}\Big)d\Gamma+\frac{1}{2}\int_{\Gamma_0}\Big[\underline{E}(\underline{n}^\alpha)\delta\underline{\sigma}_2^{\alpha}\Big]^T\cdot\Big(\overset{\frown}{\underline{u}}{}_1^{\alpha}-\overset{\frown}{\underline{u}}{}_1^{\alpha\prime}\Big)d\Gamma+\frac{1}{2}\int_{\Gamma_0}\Big[\underline{E}(\underline{n}^\alpha)\delta\underline{\sigma}^{\alpha}_3\Big]^T\Big(\overset{\frown}{\underline{u}}{}_0^{\alpha}-\overset{\frown}{\underline{u}}{}_0^{\alpha\prime}\Big)d\Gamma\Big\} \tag{30}$$

同理，可以证明有

$$\delta\Pi^{I}\equiv 0 \tag{31}$$

再将$\overset{\frown}{\underline{u}}{}_0^{\alpha}$，$\overset{\frown}{\underline{\sigma}}{}_0^{\alpha}$，$\overset{\frown}{\underline{u}}{}_1^{\alpha}$，$\overset{\frown}{\underline{\sigma}}{}_1^{\alpha}$代入（20）式中的$\Pi_4^{I}$，经变分整理后，得到

$$\delta\Pi_4^{I}=\sum_{\alpha}\int_{t_1}^{t_2}dt\Big\{\int_{\Omega_\alpha}\Big(\rho_2^\alpha\dot{\overset{\frown}{\underline{u}}}{}_0^{\alpha T}\delta\dot{\underline{u}}_2^{\alpha}+\rho_1^\alpha\dot{\overset{\frown}{\underline{u}}}{}_1^{\alpha T}\delta\dot{\underline{u}}_2+\rho_0^\alpha\dot{\underline{u}}_2^{\alpha T}\delta\dot{\underline{u}}_2^{\alpha}-\overset{\frown}{\underline{\varepsilon}}{}_0^{\alpha T}\underline{D}_2^{\alpha}\delta\underline{\varepsilon}_2^{\alpha}-\overset{\frown}{\underline{\varepsilon}}{}_1^{\alpha T}\underline{D}_1^{\alpha}\delta\underline{\varepsilon}_2^{\alpha}-\underline{\varepsilon}_2^{\alpha T}\underline{D}_0^{\alpha}\delta\underline{\varepsilon}_2^{\alpha}\Big)d\Omega+\int_{\Omega_\alpha}\underline{F}_2^{\alpha T}\delta\underline{u}_2^{\alpha}d\Omega+\int_{\Gamma_{\sigma_\alpha}}\overline{\underline{P}}_2^{\alpha T}\delta\underline{u}_2^{\alpha}d\Gamma+\frac{1}{2}\int_{\Gamma_0}\Big[\underline{E}(\underline{n}^\alpha)\delta\underline{\sigma}_2^{\alpha}\Big]^T\Big(\underline{u}_2^{\alpha}-\underline{u}_2^{\alpha\prime}\Big)d\Gamma+\frac{1}{2}\int_{\Gamma_0}\Big[\underline{E}(\underline{n}^\alpha)\underline{\sigma}_2^{\alpha}\Big]^T\cdot\Big(\delta\underline{u}_2^{\alpha}-\delta\underline{u}_2^{\alpha\prime}\Big)d\Gamma\Big\} \tag{32}$$

由$\delta\Pi_4^{I}=0$，可求得$\overset{\frown}{\underline{u}}{}_2^{\alpha}$，$\overset{\frown}{\underline{\sigma}}{}_2^{\alpha}$，其中泛函$\Pi_4^{II}$为

$$\Pi_4^{I}=\sum_{\alpha}\int_{t_1}^{t_2}\Big\{\frac{1}{2}\int_{\Omega_\alpha}\Big(\rho_0^\alpha\dot{\underline{u}}_2^{\alpha T}\dot{\underline{u}}_2^{\alpha}+2\rho_1^\alpha\dot{\underline{u}}_2^{\alpha T}\dot{\overset{\frown}{\underline{u}}}{}_1^{\alpha}+2\rho_2^\alpha\dot{\underline{u}}_2^{\alpha T}\dot{\overset{\frown}{\underline{u}}}{}_0-\underline{\varepsilon}_2^{\alpha T}\underline{D}_0^{\alpha}\underline{\varepsilon}_2^{\alpha}-$$

$$2\underline{\varepsilon_2}^{\alpha T}\underline{D_1}^{\alpha}\overset{\frown}{\underline{\varepsilon_1}}^{\alpha}-2\underline{\varepsilon_2}^{\alpha T}\underline{D_2}^{\alpha}\overset{\frown}{\underline{\varepsilon_0}}\Big)d\Omega+\int_{\Omega^{\alpha}}\underline{F_2}^{\alpha T}\underline{u_2}^{\alpha}d\Omega+\int_{\Gamma_{\sigma\alpha}}\underline{\overline{P_2}}^{\alpha T}\underline{u_2}^{\alpha}d\Gamma+$$

$$\frac{1}{2}\int_{\Gamma_0}\Big[\underline{E}(\underline{n}^{\alpha})\underline{\sigma_2}^{\alpha}\Big]^T\Big(\underline{u_2}^{\alpha}-\underline{u_2}^{\alpha\prime}\Big)d\Gamma\Big\}dt \tag{33}$$

同理可证：

$$\delta\Pi_5{}^{\rm I}=0 \tag{34}$$

如此进行下去，可得到一般形式为

$$\begin{cases}\delta\delta_{2K}{}^{\rm I}=0\\ \delta\Pi^{\rm I}{}_{2K+1}=0\quad(i=0,\ 1,\ 2,\ 3,\ \cdots)\end{cases} \tag{35}$$

其中：

$$\Pi_{2K}{}^{\rm I}=\sum_{\alpha}\int_{t_1}^{t_2}\Big\{\frac{1}{2}\int_{\Omega_\alpha}\Big(\rho_0{}^{\alpha}\underline{\dot{u}_K}^{\alpha T}\underline{\dot{u}_K}^{\alpha}+2\rho_1{}^{\alpha}\underline{\dot{u}}^{\alpha}{}_K{}^T\overset{\frown}{\underline{\dot{u}_{K-1}}}^{\alpha}+2\rho_2{}^{\alpha}\underline{\dot{u}_K}^{\alpha T}\overset{\frown}{\underline{\dot{u}_{K-2}}}^{\alpha}+\cdots+$$

$$2\rho_{K-1}{}^{\alpha}\underline{\dot{u}_K}^{\alpha}\ \overset{\frown}{\underline{\dot{u}_1}}^{\alpha T}+2\rho_K{}^{\alpha}\underline{\dot{u}_K}^{\alpha T}\overset{\frown}{\underline{\dot{u}_0}}^{\alpha}-\underline{\varepsilon_K}^{\alpha T}\underline{D_0}^{\alpha}\underline{\varepsilon_K}^{\alpha}-2\underline{\varepsilon_K}^{\alpha T}\underline{D_1}^{\alpha}\overset{\frown}{\underline{\varepsilon_{K-1}}}^{\alpha}-$$

$$2\underline{\varepsilon_K}^{\alpha T}\underline{D_2}^{\alpha}\overset{\frown}{\underline{\varepsilon_{K-2}}}^{\alpha}-\cdots-2\underline{\varepsilon_K}^{\alpha T}\underline{D_{K-1}}^{\alpha}\overset{\frown}{\underline{\varepsilon_1}}^{\alpha}-2\underline{\varepsilon_K}^{\alpha T}\underline{D_K}^{\alpha}\overset{\frown}{\underline{\varepsilon_0}}^{\alpha}\Big)d\Omega+$$

$$\int_{\Omega_\alpha}\underline{F_K}^{\alpha T}\underline{u_K}^{\alpha}d\Omega+\int_{\Gamma_{\sigma_\alpha}}\underline{\overline{P_K}}^{\alpha T}\underline{u_K}^{\alpha}d\Gamma+\frac{1}{2}\int_{\Gamma_0}\Big[\underline{E}(\underline{n}^{\alpha})\underline{\sigma_K}^{\alpha}\Big]^T\cdot$$

$$\Big(\underline{u_K}^{\alpha}-\underline{u_K}^{\alpha\prime}\Big)d\Gamma\Big\}dt \tag{36}$$

上式是弹性动力学概率分区广义变分原理的泛函，其中$\overset{\frown}{\underline{u_i}}^{\alpha}$，$\overset{\frown}{\underline{\varepsilon_i}}^{\alpha}$及$\overset{\frown}{\underline{\sigma_i}}^{\alpha}$为已求得的量。

弹性动力学概率分区广义变分原理可陈述如下：

对任意时刻t，在边界Γ_{u_α}上满足已给的位移边界条件（5）式和在区域Ω_α中满足几何关系（3）式，那么在满足时端位移条件（7）式的一切可能运动中，实际的运动$\underline{u_K}^{\alpha}$，$\underline{\varepsilon_K}^{\alpha}$和$\underline{\sigma_K}^{\alpha}$必使泛函$\Pi_{2K}{}^{\rm I}$取驻值，而$\delta\Pi_{2K+1}{}^{\rm I}=0$。

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Dynamic Region-wise Generalized Variation Principle in Probabilistic Analysis

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ABSTRACT

In this paper, the dyamic region-wise variation principle and generalized variation principle in probabilistic analysis are established, which are theoretcal basis of the probabilistic region-wise finite element method.

Key words: probabilistic variation principle, region-wise variation, generalized variation.

简　讯

蔡彪参加国家重点实验室课题研究

我校建筑声学组蔡彪副教授和同济大学王季卿教授合作的课题《半消声室声场的特性和计算机模拟》，最近获准列为中国科学院声场声信息国家重点实验室的研究课题。研究经费由该实验室资助。

该实验室支持研究现代声学的前沿新的学术思想，支持有宏大科学意义和应用前景的基础研究和开创性工作。蔡彪副教授等研究的课题符合该实验室鼓励研究的七个方向的第一个方向。

蔡彪副教授和王季卿教授前期合作的论文《Calculation of free-field deviation in an anechoic room》已于1989年在《美国声学学报》（J.Acous.Soc.Am）第85卷第3期上发表。