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三维可压缩 Euler 方程组经典解的爆破

朱旭生,李芳娥,俞银晶

(华东交通大学 基础科学学院,江西 南昌 330013)

摘要:研究了三维可压缩等熵 Euler 方程组经典解的爆破。在 Sideris T C 等研究的基础上,利用局部解具有有限传播速度的性质,通过构造适当的泛函,证明了某些初始数据较大时 Cauchy 问题的经典解必定在有限时间内爆破的结论。

关 键 词:可压缩 Euler 方程组;经典解;爆破

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关于三维可压缩 Euler 方程组[1-2]研究成果已有很多,主要集中在各种形式的弱解以及经典解的爆破[3-8],其结论是在某些指标数据较大时经典解必定在有限时间内爆破;在有阻尼的情形下获得小初值时的经典解的整体存在性,例如 Sideris TC 研究了三维可压缩欧拉方程组解的奇异性的形成,即在某些初始数据较大的情形下经典解的爆破,以及王维克等在初始数据较小时得到了带阻尼项的多维可压缩欧拉方程组经典解的整体存在性,并研究了解的点态估计等。本文继续研究三维空间中等熵 Euler 方程组的初值问题,在 Sideris TC 研究了三维空间中可压缩欧拉方程组经典解爆破的基础上,对条件进行适当的调整,结合文献[6-7],通过构造泛函,证明其经典解在有限时间内必定爆破的结论。

结论及证明

考虑三维等熵可压缩 Euler 方程组:

$$\rho_t + \mathrm{div}(\rho u) = 0 \tag{1}$$

$$\rho(u_t + u \cdot \nabla u) + \nabla p = 0 \tag{2}$$

$$p = \lambda^2 \rho^\gamma, 1 < \gamma \leqslant 3 \tag{3}$$

初始条件为:$\rho(x,0) = \rho^0(x) > 0, u(x,0) = u^0(x)$ (4)

其中:ρ, u, P 分别表示气体的密度、速度和压强;$x \in R^3$ 是空间变量,t 是时间变量,γ 是绝热指数,$t > 0, \lambda > 0$。

在有界集$\{|x| \geqslant R\}$ 外有

$$\rho^0(x) = \bar{\rho}, u^0(x) = \bar{u} = 0 \tag{5}$$

记 $\sigma = p_\rho(\bar{\rho})^{1/2} = [\lambda^2 \gamma \bar{\rho}^{\gamma-1}]^{1/2} = \lambda[\gamma \bar{\rho}^{\gamma-1}]^{1/2}, A(t) = \{x : |x| \geqslant R + \sigma t\}$,

$B(t) = \{x : |x| \leqslant R + \sigma t\}, \bar{p} = p(\bar{\rho}), \Omega(T) = \{(x,t) | x \in A(t), 0 \leqslant t < T\}$。

引理[6] 对于 $T > 0$,(ρ, u) 是(1)-(5) 式在 $\Omega(t)$ 上的经典解,则对任意的$(x,t) \in \Omega(T)$ 有$(\rho, u)(x,t) = (\bar{\rho}, 0)$。

建立数量关系式

$$F(t) = \int_{R^3} x \cdot \rho u(x,t) \mathrm{d}x, m(t) = \int_{R^3} (\rho(x,t) - \bar{\rho}) \mathrm{d}x, \eta(t) = \int_{R^3} p^{1/\gamma} - \bar{p}^{1/\gamma} \mathrm{d}x。$$

定理 如果$(\rho, u) \in C^1$ 是(1)-(5) 式的经典解,$\eta(0) \geqslant 0$,且

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作者简介:朱旭生(1968-),男,博士,副教授,研究方向为偏微分。

$$F(0) \geqslant \frac{16\pi}{3}\sigma R^4 \max\ \rho^0(x) \tag{6}$$

或

$$F(0) > \left[\int_0^{+\infty}((R+\sigma t)^2(m(0)+\bar{\rho}\,\mathrm{vol}\,B(t)))^{-1}\mathrm{d}t\right]^{-1} \tag{7}$$

则经典解必定在有限时间内爆破。

证明　假设$(\rho, u) \in C^1$是初值问题(1)－(5)式在$0 \leqslant t < T$的经典解，分部积分可得

$$m'(t) = \int_{R^3}(\rho(x,t)-\bar{\rho})_t\mathrm{d}x = \int_{R^3}\rho_t(x,t)\mathrm{d}x = -\int_{R^3}\mathrm{div}(\rho u)\mathrm{d}x = 0$$

所以

$$m(t) = m(0)$$

即

$$\int_{R^3}\rho(x,t)\mathrm{d}x = \int_{R^3}\bar{\rho}\mathrm{d}x + m(0) \tag{8}$$

又

$$\eta'(t) = \int_{R^3}\lambda^{2/\gamma}(\rho(x,t)-\bar{\rho})_t\mathrm{d}x = \lambda^{2/\gamma}m'(t) = 0$$

所以

$$\eta(t) = \eta(0)$$

即

$$\int_{R^3}\rho(x,t)\mathrm{d}x = \lambda^{\gamma/2}\eta(0) + \int_{R^3}\bar{\rho}\mathrm{d}x \tag{9}$$

同理，我们推导出

$$F'(t) = \int_{R^3}x\cdot(\rho u)_t\mathrm{d}x = \int_{R^3}x\cdot(\rho_t u - \rho u\cdot\nabla u - \nabla p)\mathrm{d}x = -\int_{R^3}x\left(u\cdot\nabla\rho u + \rho u\cdot\nabla u + \nabla p\right)\mathrm{d}x =$$

$$-\int_{R^3}\sum_{i=1}^{3}\partial_{x_i}\left[(x_i u_i)\rho u_i + x_i(p-\bar{p})\right]\mathrm{d}x + \int_{R^3}\sum_{i=1}^{3}[\rho u_i^2 + (p-\bar{p})]\mathrm{d}x = \int_{R^3}(\rho|u|^2 + 3(p-\bar{p}))\mathrm{d}x$$

利用 Jensen 不等式，可得

$$\left(\int_{B(t)}\rho^\gamma\mathrm{d}x\right)^{1/\gamma}\left(\int_{B(t)}1\mathrm{d}x\right)^{1-1/\gamma} \geqslant \int_{B(t)}\rho\mathrm{d}x$$

即

$$\left(\int_{B(t)}\rho^\gamma\mathrm{d}x\right)^{1/\gamma} \geqslant \left(\int_{B(t)}1\mathrm{d}x\right)^{1/\gamma-1}\int_{B(t)}\rho\mathrm{d}x$$

故

$$\int_{B(t)}p\mathrm{d}x = \int_{B(t)}\lambda^2\rho^\gamma\mathrm{d}x \geqslant \lambda^2\left(\int_{B(t)}1\mathrm{d}x\right)^{1-\gamma}\left(\int_{B(t)}\rho\mathrm{d}x\right)^\gamma = \lambda^2(\mathrm{vol}\,B(t))^{1-\gamma}\left(\int_{B(t)}\rho\mathrm{d}x\right)^\gamma$$

将(9)式代入上式

$$\int_{B(t)}p\mathrm{d}x \geqslant \lambda^2(\mathrm{vol}\,B(t))^{1-\gamma}\left(\lambda^{\gamma/2}\eta(0) + \int_{B(t)}\bar{\rho}\mathrm{d}x\right)^\gamma \geqslant$$

$$\lambda^2(\mathrm{vol}\,B(t))^{1-\gamma}\left(\int_{B(t)}\bar{\rho}\mathrm{d}x\right)^\gamma = \lambda^2\bar{\rho}^\gamma\mathrm{vol}\,B(t) = \mathrm{vol}\,B(t)\bar{P} = \int_{B(t)}\bar{p}\mathrm{d}x$$

所以

$$F'(t) = \int_{R^3}(\rho|u|^2 + 3(p-\bar{p}))\mathrm{d}x \geqslant \int_{R^3}\rho|u|^2\mathrm{d}x \tag{10}$$

应用 Schwartz 不等式可以得到

$$F(t)^2 = \left(\int_{B(t)}x\cdot\rho u\mathrm{d}x\right)^2 \leqslant \left(\int_{B(t)}|x|^2\rho\mathrm{d}x\right)\left(\int_{B(t)}|u|^2\rho\mathrm{d}x\right) \tag{11}$$

记$K_1(t) = \frac{4\pi}{3}(R+\sigma t)^5\max\ \rho^0(x)$，利用(8)式以及在$B(t)$上$|x| \leqslant R+\sigma t$可得

$$\int_{B(t)}|x|^2\rho\mathrm{d}x \leqslant \left(R+\sigma t\right)^2\int_{B(t)}\rho\mathrm{d}x = \left(R+\sigma t\right)^2\left(\int_{B(t)}\bar{\rho}\mathrm{d}x + m(0)\right)$$

$$=(R+\sigma t)^2\left(\int_{B(t)}\bar{\rho}\,\mathrm{d}x+\int_{B(t)}(\rho^0-\bar{\rho})\,\mathrm{d}x\right)=(R+\sigma t)^2\left(\int_{B(t)}\rho^0\,\mathrm{d}x\right)$$

$$\leqslant\frac{4\pi}{3}(R+\sigma t)^5\max\ \rho^0(x)$$

(10) －(11) 式与上式相结合可得到

$$F'(t)\geqslant K_1(t)^{-1}F(t)^2 \tag{12}$$

即

$$(-F(t)^{-1})'\geqslant K_1(t)^{-1} \tag{13}$$

将上式在[0, t] 上积分有

$$F(0)^{-1}\geqslant F(0)^{-1}-F(t)^{-1}>\left[\frac{4\pi}{3}\max\ \rho^0(x)\right]^{-1}\int_0^t(-4\sigma)^{-1}(R+\sigma\tau)'d\tau$$

$$=\left(\frac{16\pi}{3}\max\ \rho^0(x)\right)^{-1}\left[R^{-4}-(R+\sigma t)^{-4}\right]$$

也就是

$$F(0)<\frac{16\pi}{3}\max\ \rho^0(x)\left[R^{-4}-(R+\sigma T)^{-4}\right]^{-1} \tag{14}$$

如果 $T=\infty$，则令 $t\to\infty$ 时上式与(6) 式相矛盾，所以经典解不会整体存在，而是在有限时间内爆破。

下证另一条件下经典解的爆破。记 $K_2(t)=(L+\sigma t)^2(m(0)+\bar{\rho}\mathrm{vol}\,B(t))$，由定义得

$$\mathrm{vol}\,B(t)=\frac{4}{3}\pi(R+\sigma t)^3=\mathrm{vol}\,B(0)+\frac{4\pi}{3}(3R^2\sigma t+3R\sigma^2t^2+\sigma^3t^3)$$

$$m(0)+\bar{\rho}\mathrm{vol}\,B(0)=\int_{B(0)}(\rho^0-\bar{\rho})\,\mathrm{d}x+\bar{\rho}\mathrm{vol}\,B(0)=\int_{B(0)}\rho^0(x)\,\mathrm{d}x>0$$

由于 $\bar{\rho}>0$，$\mathrm{vol}\,B(t)$ 单调增加，故

$$K_2(t)=(R+\sigma t)^2(m(0)+\bar{\rho}\mathrm{vol}\,B(t))>(R+\sigma t)^2(m(0)+\bar{\rho}\mathrm{vol}\,B(0))=(R+\sigma t)^2\int_{B(0)}\rho^0\,\mathrm{d}x$$

由此易知 $\int_0^{+\infty}K_2(t)^{-1}\mathrm{d}t$ 收敛。

利用(8) 式以及在 $B(t)$ 上有 $|x|\leqslant R+\sigma t$ 可得

$$\int_{B(t)}|x|^2\rho\,\mathrm{d}x\leqslant(R+\sigma t)^2\left(\int_{B(t)}\bar{\rho}\,\mathrm{d}x+m(0)\right)=(R+\sigma t)^2(\bar{\rho}\mathrm{vol}\,B(t)+m(0))$$

(10)(11) 式与上式相结合可得

$$(-F(t)^{-1})'\geqslant K_2(t)^{-1}$$

在 $[0,t]$ 上积分

$$F(0)^{-1}-F(t)^{-1}\geqslant\int_0^tK_2^{-1}(\tau)\,\mathrm{d}\tau$$

根据(7) 式，则必存在某个时刻 T^-，当 $t\to T^-$ 时有

$$F(t)\geqslant\left(F(0)^{-1}-\int_0^tK_2^{-1}(\tau)\,\mathrm{d}\tau\right)^{-1}\to\infty$$

所以经典解不会整体存在，而是在有限时间内爆破。

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Blow-up of Classical Solution for the 3D Compressible Euler Equations

Zhu Xusheng, Li Fang'e, Yu Yinjing

(School of Basic Sciences, East China Jiaotong University, Nanchang 330013, China)

Abstract: Blow-up of classical solution for the 3D compressible Euler equations is studied in the paper. Based on Sideris T C, utilizing local solutions with finite propagation speed of the nature, by constructing suitable functional, the paper demonstrates that when initial data are larger, classical solutions of Cauchy problem must blow up in the finite time.

Key words: compressible Euler equations; classical solution; blowup

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