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一类数列的通项公式及其应用

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摘要: 给出数列 $a_1=t, a_{n+1}=\frac{pa_n+q}{ra_n+s}(n\geqslant 1)$ 的通项公式及 Fibonacci 数的一些性质。

关　键　词: 数列；通解式；Fibonacci 数

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著名数学家陈景润在著作[1]中对 Fibonacci 数作了详细研究，得出了很多结果，本文对由递推关系式：

$$a_1=t,\qquad a_{n+1}=\frac{pa_n+q}{ra_n+s}\quad n\geqslant 1 \tag{*}$$

给出的数列求得其通项公式，并由此公式得到关于 Fibonacci 数的一些性质，有的结果与 I·Tomescu 在其著作[2]中的结果不谋而合。

$$a_{n+1}=\frac{pa_n+q}{ra_n+s}=\frac{p}{r}\left(1+\frac{q/p-s/r}{a_n+s/r}\right),\quad 令\ \overline{U}_{n+1}=a_{n+1}+s/r,$$

因为

$$a=\frac{p+s}{r},\qquad b=\frac{p}{r}\left(\frac{q}{p}-\frac{s}{r}\right),则(*)式变为: \tag{**}$$

$$\overline{U}_1=c,\quad \overline{U}_{n+1}=a+\frac{b}{U_n}\quad n\geqslant 1\qquad (a\neq 0, b\neq 0, c\neq 0)$$

定理 1　由递推式(**)确定的数列的通式为(规定：$\sum_{j=1}^{k}f(j)=0, k<0$)

$$\overline{U}_n=\begin{cases}\dfrac{c\sum\limits_{j=0}^{[\frac{2k}{2}]-1}C_{2k-1-j}^{j}a^{2k-1-2j}b^{j}+b\sum\limits_{j=0}^{[\frac{2k}{2}]-1}C_{2k-2-j}^{j}a^{2k-2-2j}b^{j}}{c\sum\limits_{j=0}^{[\frac{2k}{2}]-1}C_{2k-2-j}^{j}a^{2k-2-2j}b^{j}+b\sum\limits_{j=0}^{[\frac{2k}{2}]-2}C_{2k-3-j}^{j}a^{2k-3-2j}b^{j}} & n=2k,\quad k=1,2,\cdots, \quad (1)\\[3ex] \dfrac{c\sum\limits_{j=0}^{[\frac{2k+1}{2}]}C_{2k-j}^{j}a^{2k-2j}b^{j}+b\sum\limits_{j=0}^{[\frac{2k+1}{2}]-1}C_{2k-1-j}^{j}a^{2k-1-2j}b^{j}}{c\sum\limits_{j=0}^{[\frac{2k+1}{2}]-1}C_{2k-1-j}^{j}a^{2k-1-2j}b^{j}+b\sum\limits_{j=0}^{[\frac{2k+1}{2}]-1}C_{2k-2-j}^{j}a^{2k-2-2j}b^{j}} & n=2k+1,\quad k=1,2,\cdots, \quad (2)\end{cases}$$

证　$k=1$ 时

由(1)得　$\overline{U}_2=\frac{ca+b}{c}=a+\frac{b}{U_1}$，　结论成立；

由(2)得　$\overline{U}_3=\frac{c(a^2+b)+ba}{ca+b}=a+\frac{b}{U_2}$，　结论成立。

假设 $k\leqslant m-1$ 时结论成立，即

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$$\overline{U}_{2m-1}=\overline{U}_{2(m-1)+1}=\frac{c\sum_{j=0}^{[\frac{2m-1}{2}]}C_{2m-2-j}^{j}a^{2m-2-2j}b^{j}+b\sum_{j=0}^{[\frac{2m-1}{2}]-1}C_{2m-3-j}^{j}a^{2m-3-2j}b^{j}}{c\sum_{j=0}^{[\frac{2m-1}{2}]-1}C_{2m-3-j}^{j}a^{2m-3-2j}b^{j}+b\sum_{j=0}^{[\frac{2m-1}{2}]-1}C_{2m-4-j}^{j}a^{2m-4-2j}b^{j}}$$

下面证明 $k=m$ 时结论也成立⒀

因为 $\overline{U}_{2m}=a+\frac{b}{\overline{U}_{2m-1}}=\frac{a\overline{U}_{2m-1}+b}{\overline{U}_{2m-1}}$，　化简以后，其分母为

$$c\sum_{j=0}^{[\frac{2m-1}{2}]}C_{2m-2-j}^{j}a^{2m-2-2j}b^{j}+b\sum_{j=0}^{[\frac{2m-1}{2}]-1}C_{2m-3-j}^{j}a^{2m-3-2j}b^{j} \tag{3}$$

化简后的分子为

$$\begin{aligned}
&c\sum_{j=0}^{[\frac{2m-1}{2}]}(C_{2m-2-j}^{j}a^{2m-2-2j}b^{j})a+b\sum_{j=0}^{[\frac{2m-1}{2}]-1}(C_{2m-3-j}^{j}a^{2m-3-2j}b^{j})a\\
&+c\sum_{j=0}^{[\frac{2m-1}{2}]-1}(C_{2m-3-j}^{j}a^{2m-3-2j}b^{j})b+b\sum_{j=0}^{[\frac{2m-1}{2}]-1}(C_{2m-4-j}^{j}a^{2m-4-2j}b^{j})b\\
&=c\Big(a^{2m-1}+C_{2m-3}^{1}a^{2m-3}b+C_{2m-4}^{2}a^{2m-5}b^{2}+C_{2m-5}^{3}a^{2m-7}b^{3}+\cdots\\
&+C_{2m-2-[\frac{2m-1}{2}]}^{[\frac{2m-1}{2}]}a^{2m-1-2[\frac{2m-1}{2}]}b^{[\frac{2m-1}{2}]}+a^{2m-3}b+C_{2m-4}^{1}a^{2m-5}b^{2}+C_{2m-5}^{2}a^{2m-7}b^{3}+\cdots\\
&+C_{2m-3-([\frac{2m-1}{2}]-1)}^{[\frac{2m-1}{2}]-1}a^{2m-3-2([\frac{2m-1}{2}]-1)}b^{[\frac{2m-1}{2}]-1}\Big)+b\Big(a^{2m-2}+C_{2m-4}^{1}a^{2m-4}b+C_{2m-5}^{2}a^{2m-6}b^{2}+\cdots\\
&+C_{2m-3-([\frac{2m-1}{2}]-1)}^{[\frac{2m-1}{2}]-1}a^{2m-2-2([\frac{2m-1}{2}]-1)}b^{[\frac{2m-1}{2}]-1}+a^{2m-4}b+C_{2m-5}^{1}a^{2m-6}b^{2}+C_{2m-6}^{2}a^{2m-8}b^{3}+\cdots\\
&+C_{2m-4-([\frac{2m-1}{2}]-1)}^{[\frac{2m-1}{2}]-1}a^{2m-4-2([\frac{2m-1}{2}]-1)}b^{[\frac{2m-1}{2}]-1}\Big)\\
&=c\Big(a^{2m-1}+C_{2m-2}^{1}a^{2m-3}b+C_{2m-3}^{2}a^{2m-5}b^{2}+\cdots+C_{2m-2-([\frac{2m}{2}]-2)}^{[\frac{2m}{2}]-1}a^{2m-1-2([\frac{2m}{2}]-1)}b^{[\frac{2m}{2}]-1}\Big)\\
&+b\Big(a^{2m-2}+C_{2m-3}^{1}a^{2m-4}b+C_{2m-4}^{2}a^{2m-6}b^{2}+\cdots\\
&+C_{2m-3-([\frac{2m}{2}]-2)}^{[\frac{2m-1}{2}]-1}a^{2m-2-2([\frac{2m-1}{2}]-1)}b^{[\frac{2m-1}{2}]-1}+a^{2m-4-2([\frac{2m}{2}]-2)}b^{[\frac{2m}{2}]-1}\Big)\\
&=c\sum_{j=0}^{[\frac{2m}{2}]-1}C_{2m-1-j}^{j}a^{2m-1-2j}b^{j}+b\sum_{j=0}^{[\frac{2m}{2}]-1}C_{2m-2-j}^{j}a^{2m-2-2j}b^{j}
\end{aligned} \tag{4}$$

由(3)、(4) 式可知(1) 式成立⒀

利用(1) 很易证得(2) 式成立⒀　　(毕)

下面给出(1)、(2) 两式的应用⒀

考虑递推关系：$\overline{U}_1=1,\quad \overline{U}_{n+1}=1+\frac{1}{\overline{U}_n},\quad n\geqslant 1$

由(1)(2) 两式得　(规定：$k<0$ 时，$\sum_{j=0}^{k}f(j)=0$)

$$\overline{U}_n=\begin{cases}\left(\sum_{j=0}^{[\frac{2k}{2}]-1}C_{2k-1-j}^{j}+\sum_{j=0}^{[\frac{2k}{2}]-1}C_{2k-2-j}^{j}\right)\Big/\left(\sum_{j=0}^{[\frac{2k}{2}]-1}C_{2k-2-j}^{j}+\sum_{j=0}^{[\frac{2k}{2}]-2}C_{2k-3-j}^{j}\right) & n=2k,\ k=1,2\cdots\\ \left(\sum_{j=0}^{[\frac{2k+1}{2}]}C_{2k-j}^{j}+\sum_{j=0}^{[\frac{2k+1}{2}]-1}C_{2k-1-j}^{j}\right)\Big/\left(\sum_{j=0}^{[\frac{2k+1}{2}]-1}C_{2k-1-j}^{j}+\sum_{j=0}^{[\frac{2k+1}{2}]-1}C_{2k-2-j}^{j}\right) & n=2k+1,\ k=1,2\cdots\end{cases}$$

另一方面，由文[3] 中的方法可得

$$\overline{U}_n=\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)\Big/\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n}-\left(\frac{1-\sqrt{5}}{2}\right)^{n}\right)=F_n/F_{n-1}$$

$n=2k+1$ 时得

$$\left(\sum_{j=0}^{k} C_{2k-j}^{j} + \sum_{j=0}^{k-1} C_{2k-1-j}^{j}\right) F_{2k} = \left(\sum_{j=0}^{k-1} C_{2k-1-j}^{j} + \sum_{j=0}^{k-1} C_{2k-2-j}^{j}\right) F_{2k+1}$$

$$\Rightarrow \forall k \in N \quad \left(1 + \sum_{j=0}^{k-1} C_{2k-j}^{j+1}\right) F_{2k} = \left(1 + \sum_{j=0}^{k-1} C_{2k-j}^{j}\right) F_{2k+1} \tag{5}$$

$$\Rightarrow \forall k \in N \quad F_{2k} = 1 + \sum_{j=0}^{k-1} C_{2k-j}^{j}, \quad F_{2k+1} = 1 + \sum_{j=0}^{k-1} C_{2k-j}^{j+1}$$

这与 I·Tomescu 在文献[2]中对 Fibonacci 数给出的定义不谋而合[13]

下面利用(5)式给出 Fibonacci 数的一些性质[13]

定理 2 $F_{2k} = F_{k-2}F_k + F_{k-1}F_{k+1}, \quad F_{2k+1} = F_{k-1}F_k + F_kF_{k+1} \qquad k \in N$

证 记 $\sum_{j=0}^{k-1} C_{2k-j}^{j+1} = A$，$\sum_{j=0}^{k-1} C_{2k-j}^{j} = B$，由(5)式

一方面得 $$F_{2k-1} - AF_{2k} + BF_{2k+1} = 0 \tag{6}$$

另一方面得

$$1 + A = (1 + B)\frac{F_{2k+1}}{F_{2k}} = 1 + B + (1 + B)\frac{F_{2k-1}}{F_{2k}}$$

$$\Rightarrow 1 + B = (A - B)\frac{F_{2k}}{F_{2k-1}} = A - B + (A - B)\frac{F_{2k-2}}{F_{2k-1}}$$

$$\Rightarrow (F_0 - F_0A + F_2B)\frac{F_{2k-1}}{F_{2k-2}} = A - B$$

$$\Rightarrow A - B = F_0 - F_0A + F_2B + (F_0 - F_0A + F_2B)\frac{F_{2k-3}}{F_{2k-2}}$$

$$\Rightarrow F_0 - F_0A + F_2B = -F_0 + F_2A - F_3B + (-F_0 + F_2A - F_3B)\frac{F_{2k+4}}{F_{2k-3}}$$

$$\Rightarrow (F_2 - F_3A + F_4B)\frac{F_{2k-3}}{F_{2k-4}} = -F_0 + F_2A - F_3B$$

$$\Rightarrow -F_1 + F_2A - F_3B = F_2 - F_3A + F_4B + (F_2 - F_3A + F_4B)\frac{F_{2k-5}}{F_{2k-4}}$$

$$\Rightarrow (-F_3 + F_4A - F_5B)\frac{F_{2k-4}}{F_{2k-5}} = F_2 - F_3A + F_4B$$

即 $$(F_3 - F_4A + F_5B)\frac{F_{2k-4}}{F_{2k-5}} = -F_2 + F_3A - F_4B$$

$$\Rightarrow \cdots\cdots$$

$$\Rightarrow (F_{k-2} - F_{k-1}A + F_kB)\frac{F_{2k-(k-1)}}{F_{2k-k}} = -F_{k-3} + F_{k-2}A - F_{k-1}B$$

$$\Rightarrow (F_{k-3}F_k + F_{k-2}F_{k+1}) - (F_{k-2}F_k + F_{k-1}F_{k+1})A + (F_{k-1}F_k + F_kF_{k+1})B = 0 \tag{7}$$

由(6)、(7)得

$$F_{2k} = F_{k-2}F_k + F_{k-1}F_{k+1};\ F_{2k+1} = F_{k-1}F_k + F_kF_{k+1} \qquad \text{(毕)}$$

由定理 2 的结果立即可得

定理 3 $\forall k \in N, \quad F_k \mid F_{2k+1}, \quad (F_{k-1} + F_{k+1}) \mid F_{2k+1}$

且 $F_{2k+2} = F_k^2 + F_{k+1}^2 \qquad k = 0, 1, 2\cdots$

引理 $\forall k \in N, \quad F_{k+1}F_{k-1} - F_k^2 = (-1)^{k+1}$

下面的证明是 I·Tomescu 的[13]

证 因为 $$\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}^{1+1} = \begin{pmatrix}2 & 1\\1 & 1\end{pmatrix} = \begin{pmatrix}F_2 & F_1\\F_1 & F_0\end{pmatrix}$$

假设 $$\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}^{m+1} = \begin{pmatrix}F_{m+1} & F_m\\F_m & F_{m-1}\end{pmatrix}$$

则 $$\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}^{(m+1)+1} = \begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}^{m+1}\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix} = \begin{pmatrix}F_{m+1} & F_m\\F_m & F_{m-1}\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix} = \begin{pmatrix}F_{m+2} & F_{m+1}\\F_{m+1} & F_{(m+1)-1}\end{pmatrix}$$

所以 $\forall k \in N$ 有 $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{k+1} = \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix}$,两边取行列式立即得证结论⒀

定理 4 $F_{k-3}F_k - F_{k-1}F_{k-2} = (-1)^{k+1}$, $k = 3, 4, \cdots$

证

$$F_{k-3}F_k - F_{k-1}F_{k-2} = (F_{k-1} - F_{k-2})(F_{k-1} + F_{k-2}) - F_{k-1}F_{k-2}$$
$$= F_{k-1}^2 - F_{k-2}^2 - F_{k-1}F_{k-2} = F_{k-1}^2 - F_{k-2}F_k = -(F_{(k-1)-1}F_{(k-1)+1} - F_{k-1}^2)$$
$$\overset{\text{引理}}{=} (-1)(-1)^{(k-1)+1} = (-1)^{k+1} \qquad \text{(毕)}$$

定理 5 $(F_k + F_{k-3}) \mid F_{k-1}(F_{k+1} + F_{k-2})$　$k = 3, 4, \cdots$

证 由引理及定理 4 得

$$F_k(F_k + F_{k-3}) = F_{k-1}(F_{k+1} + F_{k-2})$$

所以 $(F_k + F_{k-3}) \mid F_{k-1}(F_{k+1} + F_{k-2})$　(毕)

定理 6 $\forall k \in N$　$(F_{2k} - (-1)^k) \mid F_{k-1}F_{2k+1}$

证 由定理 2 得

$$F_kF_{2k} = F_{k-2}F_k^2 + F_{k-1}F_{k+1} + F_k$$
$$F_{k-1}F_{2k+1} = F_kF_{k-1}^2 + F_kF_{k-1}F_{k+1}$$

上面两式相减得

$$F_kF_{2k} - F_{k-1}F_{2k+1} = (F_{k-2}F_k - F_{k-1}^2)F_k \overset{\text{引理}}{=} (-1)^kF_k$$
$$\Rightarrow F_k(F_{2k} - (-1)^k) = F_{k-1}F_{2k+1}$$

所以 $(F_{2k} - (-1)^k) \mid F_{k-1}F_{2k+1}$　(毕)

定理 7 $F_{2k+1}F_{k-2} - F_{2k}F_{k-1} = (-1)^kF_{k+1}$　$k = 2, 3, \cdots$

证 由定理 2 及引理得

$$F_{2k+1}F_{k-2} - F_{2k}F_{k-1} = (F_{k-2}F_{k-1}F_k + F_{k-2}F_kF_{k+1}) - (F_{k-1}F_{k-2}F_k + F_{k-1}^2F_{k+1})$$
$$= (F_{k-2}F_k - F_{k-1}^2)F_{k+1} = (-1)^kF_{k+1} \qquad \text{(毕)}$$

(1)、(2) 两式还有许多关于恒等式建立方面的应用,我们另文再述⒀

参考文献:

[1] 陈景润⒆组合数学简介[M]⒆天津:天津科学技术出版社,1983⒆

[2] I Tomescu· 组合学引论[M]⒆北京:高等教育出版社,1988⒆

[3] 王江云⒆关于《通项方程的一种解法》一文的注记[J]⒆数学通报,1999,(6):33⒆

The General Expression of the sequence Defined by $\overline{U}_{n+1} = a + \frac{b}{U_n}$ and Its Application

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Abstract: We get the general expression of the sequence defined by $\overline{U}_{n+1} = a + \frac{b}{U_n}$ $n \geqslant 1$, $\overline{U}_1 = c$ and some propositions of Fibonacci number·

Key words: sequence; general expression; Fibonacci number