第9卷 第1期 华东交通大学学报 Vol.9 No.1
1992年 3月 Journal of East China Jiaotong University Mar. 1992

多元复合函数求导方法

朱静萍 龚代华

（江西财经学院） （基础课部）

摘　要

本文针对复合函数导数教学中的难点，提出了两种新的求导方法。

关键词： 复合函数；中间变量；求导方法

多元复合函数的导数是高等数学中的一个难点，学生求导时常会出错。问题并不在于复合函数导数公式过分复杂，实际上求导公式相当简单。那么，原因是什么呢？通过这几年的教学实践，我们观察到学生对$Z=f[u(x,y),v(x,y)]$，$z=f[u(x,y),v(x,y),w(x,y)]$这类能直接应用公式求导的题目做起来得心应手，甚至推广到$z=f[u_1(x,y),u_2(x,y),\cdots,u_n(x,y)]$也做得出来。而对象$z=f[x,\varphi(x,y)]$这样简单的函数的导数却往往不会动手。其原因在于，对$z=f[x,\varphi(x,y)]$中的x不知如何处理，是作为中间变量呢？还是自变量呢？为了解决这个问题，我们在教学中采取如下方法。

先给出推广的复合函数求导公式。

定理[1] 设有复合函数$A=f[u_1(x,y,z),u_2(x,y,z),\cdots,u_n(x,y,z)]$，函数$f(u_1,u_2,\cdots,u_n)$有连续偏导数$\frac{\partial f}{\partial u_1}$，$\frac{\partial f}{\partial u_2}$，$\cdots$，$\frac{\partial f}{\partial u_n}$，$u_1(x,y,z)$，$u_2(x,y,z)$，$\cdots$，$u_n(x,y,z)$的三个偏导数都存在，则复合函数可导且

$$\frac{\partial A}{\partial x}=\frac{\partial f}{\partial u_1}\frac{\partial u_1}{\partial x}+\frac{\partial f}{\partial u_2}\frac{\partial u_2}{\partial x}+\cdots+\frac{\partial f}{\partial u_n}\frac{\partial u_n}{\partial x} \quad (1)$$

$$\frac{\partial A}{\partial y}=\frac{\partial f}{\partial u_1}\frac{\partial u_1}{\partial y}+\frac{\partial f}{\partial u_2}\frac{\partial u_2}{\partial y}+\cdots+\frac{\partial f}{\partial u_n}\frac{\partial u_n}{\partial y} \quad (2)$$

$$\frac{\partial A}{\partial z}=\frac{\partial f}{\partial u_1}\frac{\partial u_1}{\partial z}+\frac{\partial f}{\partial u_2}\frac{\partial u_2}{\partial z}+\cdots+\frac{\partial f}{\partial u_n}\frac{\partial u_n}{\partial z} \quad (3)$$

利用（1），（2），（3）式可解决以下两种函数的求导问题。

1．$z=f[x,y,R(x,y),S(x,y)]$令$u=u(x,y)$，$v=v(x,y)$，$R=R(x,y)$，$S=S(x,y)$则u，v，R，S为中间变量，x，y为自变量，可直接用公式求导。

本文于1991年10月18日收到

2．$u=f[x, y, z, \varphi(x, y, z)]$　令$p=p(x, y, z)=x$，$q=q(x, y, z)$，$l=l(x, y, z)$，$m=m(x, y, z)$．

则中间变量、自变量一清二楚。

利用上述扩充引进中间变量的方法，使求复合函数的导数变得更容易且不易出错，经实践证明效果较好。

对于求$z=f[u(x, y), v(x, y)]$的二阶导数，学生最容易忽视 $\frac{\partial f}{\partial u}$，$\frac{\partial f}{\partial v}$ 仍是以u，v为中间变量的复合函数这个事实，故求导时往往会少许多项，为了避免这种情况发生，我们采取如下办法。

设有复合函数$z=f(u_1, u_2, \cdots, u_n)$，$u_1=u_1(x_1, x_2, \cdots, x_m)$，$u_2=u_2(x_1, x_2, \cdots, x_m)$，…，$u_n=u_n(x_1, x_2, \cdots, x_m)$，这里$f(u_1, u_2, \cdots, u_n)$有二阶连续偏导数，$u_1(x_1, x_2, \cdots, x_m)$，$u_2(x_1, x_2, \cdots, x_m)$，…，$u_n(x_1, x_2, \cdots, x_m)$有二阶连续偏导数。

由复合函数求导公式（1）（2）（3）有

$$\frac{\partial z}{\partial x_i}=\frac{\partial f}{\partial u_1}\frac{\partial u_1}{\partial x_i}+\frac{\partial f}{\partial u_2}\frac{\partial u_2}{\partial x_i}+\cdots+\frac{\partial f}{\partial u_n}\frac{\partial u_n}{\partial x_i} \qquad (4)$$

$$(i=1, 2, \cdots, m)$$

为简便记：$f_i=\frac{\partial f}{\partial u_i}$　$(i=1, 2, \cdots, n)$ $f_{ij}=\frac{\partial^2 f}{\partial u_i \partial u_j}(i=1, 2, \cdots, n, j=1, 2, \cdots, n)$于是（4）式可写为

$$\frac{\partial z}{\partial x_i}=f_1\frac{\partial u_1}{\partial x_i}+f_2\frac{\partial u_2}{\partial x_i}+\cdots+f_n\frac{\partial u_n}{\partial x_i}$$

再对x_j求导得

$$\frac{\partial^2 z}{\partial x_i \partial x_j}=\frac{\partial}{\partial x_j}\left(\frac{\partial z}{\partial x_i}\right)=\frac{\partial}{\partial x_j}\left(f_1\frac{\partial u_1}{\partial x_i}+f_2\frac{\partial u_2}{\partial x_i}+\cdots+f_n\frac{\partial u_n}{\partial x_i}\right)$$

$$=\frac{\partial}{\partial x_j}(f_1)\frac{\partial u_1}{\partial x_i}+\frac{\partial}{\partial x_j}(f_2)\frac{\partial u_2}{\partial x_i}+\cdots+\frac{\partial}{\partial x_j}(f_n)\frac{\partial u_n}{\partial x_i}$$

$$+f_1\frac{\partial^2 u_1}{\partial x_i \partial x_j}+f_2\frac{\partial^2 u_2}{\partial x_i \partial x_j}+\cdots+f_n\frac{\partial^2 u_n}{\partial x_i \partial x_j}$$

$$=\left(f_{11}\frac{\partial u_1}{\partial x_j}+f_{12}\frac{\partial u_2}{\partial x_j}+\cdots+f_{1n}\frac{\partial u_n}{\partial x_j}\right)\frac{\partial u_1}{\partial x_i}$$

$$+\left(f_{21}\frac{\partial u_1}{\partial x_j}+f_{22}\frac{\partial u_2}{\partial x_j}+\cdots+f_{2n}\frac{\partial u_n}{\partial x_j}\right)\frac{\partial u_2}{\partial x_i}$$

$$+\cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots$$

$$+\left(f_{n1}\frac{\partial u_1}{\partial x_j}+f_{n2}\frac{\partial u_2}{\partial x_j}+\cdots+f_{nn}\frac{\partial u_n}{\partial x_j}\right)\frac{\partial u_n}{\partial x_i}$$

$$+f_1\frac{\partial^2 u_1}{\partial x_i\partial x_j}+f_2\frac{\partial^2 u_2}{\partial x_i\partial x_j}+\cdots+f_n\frac{\partial^2 u_n}{\partial x_i x_j} \qquad (5)$$

$$=\left(\frac{\partial u_1}{\partial x_i}\ \frac{\partial u_2}{\partial x_i}\ \cdots\ \frac{\partial u_n}{\partial x_i}\right)\begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{pmatrix}\begin{pmatrix} \frac{\partial u_1}{\partial x_j} \\ \frac{\partial u_2}{\partial x_j} \\ \vdots \\ \frac{\partial u_n}{\partial x_j} \end{pmatrix}$$

$$+f_1\frac{\partial^2 u_1}{\partial x_i\partial x_j}+f_2\frac{\partial^2 u_2}{\partial x_i\partial x_j}+\cdots+f_n\frac{\partial^2 u_n}{\partial x_i\partial x_j} \qquad (6)$$

由于$f(u_1, u_2, \cdots, u_n)$具有二阶连续偏导数，故$\frac{\partial^2 f}{\partial u_i\partial u_j}=\frac{\partial^2 f}{\partial u_j\partial u_i}$即$f_{ij}=f_{ji}$（$i=1, 2, \cdots, n, j=1, 2, \cdots, n$），因此(5)中的矩阵$\begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{pmatrix}$是一个对称矩阵。

（5）式中的后半部分亦可用矩阵表示为

$$(f_1\quad f_2\quad \cdots\quad f_n)\begin{pmatrix} \frac{\partial^2 u_1}{\partial x_i\partial x_j} \\ \frac{\partial^2 u_2}{\partial x_i\partial x_j} \\ \vdots \\ \frac{\partial^2 u_n}{\partial x_i\partial x_j} \end{pmatrix}$$

利用上述方法求多元复合函数的导数，只要分别求出矩阵的每一项，这些一般只要作简单的求导运算即可求得。对于$z=f[x, y, u(x, y), v(x, y)]$的二阶导数，可以先采取扩充引进中间变量的方法，再利用公式（6）。

例1． 求$z=f(x, y, e^{xy}, x^3y^2)$的二阶导数$\frac{\partial^2 z}{\partial x^2}$，$\frac{\partial^2 z}{\partial x\partial y}$

解：令$u_1=x$，$u_2=y$，$u_3=e^{xy}$，$u_4=x^3y^2$则

$$\frac{\partial u_1}{\partial x}=1,\quad \frac{\partial u_2}{\partial x}=0,\quad \frac{\partial u_3}{\partial x}=ye^{xy},\quad \frac{\partial u_4}{\partial x}=3x^2y^2$$

$$\frac{\partial u_1}{\partial y}=0, \quad \frac{\partial u_2}{\partial y}=1, \quad \frac{\partial u_3}{\partial y}=xe^{xy}, \quad \frac{\partial u_4}{\partial y}=2x^3y$$

$$\frac{\partial^2 u_1}{\partial x^2}=0, \quad \frac{\partial^2 u_2}{\partial x^2}=0, \quad \frac{\partial^2 u_3}{\partial x^2}=y^2e^{xy}, \quad \frac{\partial^2 u_4}{\partial x^2}=6xy^2$$

$$\frac{\partial^2 u_1}{\partial x\partial y}=0, \frac{\partial^2 u_2}{\partial x\partial y}=0, \frac{\partial^2 u_3}{\partial x\partial y}=(1+xy)e^{xy}, \quad \frac{\partial^2 u_4}{\partial x\partial y}=6x^2y$$

故
$$\frac{\partial^2 z}{\partial x^2}=\left(\frac{\partial u_1}{\partial x} \quad \frac{\partial u_2}{\partial x} \quad \frac{\partial u_3}{\partial x} \quad \frac{\partial u_4}{\partial x}\right)\begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{41} & f_{42} & f_{43} & f_{44} \end{pmatrix}\begin{pmatrix} \frac{\partial u_1}{\partial x} \\ \frac{\partial u_2}{\partial x} \\ \frac{\partial u_3}{\partial x} \\ \frac{\partial u_4}{\partial x} \end{pmatrix}$$

$$+f_1\frac{\partial^2 u_1}{\partial x^2}+f_2\frac{\partial^2 u_2}{\partial x^2}+f_3\frac{\partial^2 u_3}{\partial x^2}+f_4\frac{\partial^2 u_4}{\partial x^2}$$

$$=(1 \quad 0 \quad ye^{xy} \quad 3x^2y^2)\begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{41} & f_{42} & f_{43} & f_{44} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ ye^{xy} \\ 3x^2y^2 \end{pmatrix}$$

$$+y^2e^{xy}f_3+6xy^2f_4$$

$$=f_{11}+y^2e^{2xy}f_{33}+9x^4y^4f_{44}+2ye^{xy}f_{13}+6x^2y^2f_{14}+6x^2y^3e^{xy}f_{34}+y^2e^{xy}f_3+6xy^2f_4$$

$$\frac{\partial^2 z}{\partial x\partial y}=\left(\frac{\partial u_1}{\partial x} \quad \frac{\partial u_2}{\partial x} \quad \frac{\partial u_3}{\partial x} \quad \frac{\partial u_4}{\partial x}\right)\begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{41} & f_{42} & f_{43} & f_{44} \end{pmatrix}\begin{pmatrix} \frac{\partial u_1}{\partial y} \\ \frac{\partial u_2}{\partial y} \\ \frac{\partial u_3}{\partial y} \\ \frac{\partial u_4}{\partial y} \end{pmatrix}$$

$$=(1 \quad 0 \quad ye^{xy} \quad 3x^2y^2)\begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{41} & f_{42} & f_{43} & f_{44} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ xe^{xy} \\ 2x^3y \end{pmatrix}$$

$$+(1+xy)e^{xy}f_3+6x^2yf_4$$

$$=xye^{2xy}f_{33}+6x^5y^3f_{44}+f_{12}+xe^{xy}f_{13}+2x^3yf_{14}+ye^{xy}f_{32}+x^2y^2(3x+2y)f_{34}$$

$$+3x^2y^2f_{42}+(1+xy)e^{xy}f_3+6x^2yf_4$$

例2. 设$z=f(u,v)$在任意点(u,v)有二阶连续偏导数，又$u=xy$，$v=\dfrac{x^2-y^2}{2}$

求 $\dfrac{\partial^2 z}{\partial x^2}$，$\dfrac{\partial^2 z}{\partial y^2}$

解：记 $f_{11}=f_{uu}$　$f_{12}=f_{21}=f_{uv}$，　$f_{22}=f_{vv}$

$$\frac{\partial u}{\partial x}=y,\quad \frac{\partial v}{\partial x}=x;\quad \frac{\partial u}{\partial y}=x,\quad \frac{\partial v}{\partial y}=-y;$$

故
$$\frac{\partial^2 z}{\partial x^2}=\left(\frac{\partial u}{\partial x}\quad \frac{\partial v}{\partial x}\right)\begin{pmatrix} f_{11} & f_{12}\\ f_{21} & f_{22}\end{pmatrix}\begin{pmatrix}\dfrac{\partial u}{\partial x}\\ \dfrac{\partial v}{\partial x}\end{pmatrix}+f_1\frac{\partial^2 u}{\partial x^2}+f_2\frac{\partial^2 u}{\partial x^2}$$

$$=(y\quad x)\begin{pmatrix} f_{11} & f_{12}\\ f_{21} & f_{22}\end{pmatrix}\begin{pmatrix} y\\ x\end{pmatrix}+f_2$$

$$=y^2f_{11}+2xyf_{12}+y^2f_{22}+f_2$$

$$\frac{\partial^2 z}{\partial y^2}=\left(\frac{\partial u}{\partial y}\quad \frac{\partial v}{\partial y}\right)\begin{pmatrix} f_{11} & f_{12}\\ f_{21} & f_{22}\end{pmatrix}\begin{pmatrix}\dfrac{\partial u}{\partial y}\\ \dfrac{\partial v}{\partial y}\end{pmatrix}+f_1\frac{\partial^2 u}{\partial y^2}+f_2\frac{\partial^2 v}{\partial y^2}$$

$$=(x\quad -y)\begin{pmatrix} f_{11} & f_{12}\\ f_{21} & f_{22}\end{pmatrix}\begin{pmatrix} x\\ -y\end{pmatrix}-f_2$$

$$=x^2f_{11}-2xyf_{12}+y^2f_{22}-f_2$$

参 考 文 献

〔1〕 吉林大学·数学分析（下）·北京：人民教育出版社，1979：52—53

An Expression of Derivation for Composite Function of Many Variable

Zhu Jingping Gong Daihua

Abstract

In light of student's difficulty in studying derivation for composite function of many variable, This paper gives out an expression of derivation.

Key words: composite function; middle variable; method of derivation