第13卷第1期　　华东交通大学学报　　Vol. 13 No. 1
1996年3月　　Journal of East China Jiaotong University　　Mar. 1996

也论一个组合数学问题

周学松

（基础课部）

摘　要　文献〔1〕中讨论了展开式$(\eta_1+b\eta_0)(\eta_2+b\eta_1+b^2\eta_0)\cdots(\eta_n+b\eta_{n-1}+\cdots+b^n\eta_0)=\sum_{j=0}^{C_2^{n+1}} b^j\cdot A_j(n)$，引入了一个求$A_j(n)$里所有可能的项$\eta_{i_1}^{j_1}\cdot\eta_{i_2}^{j_2}\cdots\eta_{i_m}^{j_m}$的规则，并对所有这样的项的系数给出公式：$C_{j_m}^{n-i_m+1}\cdot C_{j_{m-1}}^{n-i_{m-1}-j_m+1}\cdots C_{j_{m-t}}^{n-i_{m-t}-\sum_{i=m-t+1}^{m}j_i+1}\cdots C_{j_2}^{n-i_2-\sum_{i=3}^{m}j_i+1}$. 本文讨论了同一展开式，对$A_j(n)$给出了一个直接快速计算公式：$A_j(n)=\sum_{\substack{(i_1,i_2,\cdots,i_n)\\ j=\sum_{u=1}^{n} i_u\\ 0\leqslant i_u\leqslant u}}\left(\prod_{t=1}^{n}\eta_{t-i_t}\right)$.

关键词　组合论，多项式展开，算子

分类号　O157.5

0　引　言

文献〔1〕中指出，在时间序列研究中，出现如下组合数学问题（可视b为系数，η_K为变数）：

$$(\eta_1+b\eta_0)(\eta_2+b\eta_1+b^2\eta_0)\cdots(\eta_n+b\eta_{n-1}+\cdots+b^n\eta_0)=\sum_{j=0}^{\frac{1}{2}n(n+1)} b^j\cdot A_j(n) \tag{1}$$

易见，$A_j(n)$为$\eta_0,\eta_1,\cdots,\eta_n$的正整系数$n$次齐次多项式，问除了用多项式展开慢慢算，是否有更简单的巧妙公式. 文献〔1〕中引进了一个算子D，D作用于$A_{K+1}(n)$，可得到$A_K(n)$中所有的项，然后对$A_K(n)$中的每一项的系数给出一个公式

$$C_{j_m}^{n-i_m+1}\cdot C_{j_{m-1}}^{n-i_{m-1}-j_m+1}\cdots C_{j_{m-t}}^{n-i_{m-t}-\sum_{i=m-t+1}^{m}j_i+1}\cdots C_{j_2}^{n-j_2-\sum_{i=3}^{m}j_i+1}$$

本文讨论了同一问题，对$A_j(n)$给出了一个直接计算公式.

收稿日期：1995-05-20.　　周学松，男，1956年生，副教授.

1 主要结果及证明

为叙述方便，记 $V^i_j=V^i\eta_j=\eta_{j-i}$ $(j\geqslant i)$； $j<i$ 时 V^i_j 无意义.

$$V^j(123\cdots m)=V^j(\eta_1\eta_2\cdots\eta_m)=\sum_{\substack{(i_1,i_2,\cdots,i_m)\\ j=\sum_{u=1}^{m} i_u\\ 0\leqslant i_u\leqslant u}} V^{i_1}_1V^{i_2}_2\cdots V^{i_m}_m=\sum_{\substack{(i_1,i_2,\cdots,i_m)\\ j=\sum_{u=1}^{m} i_u\\ 0\leqslant i_u\leqslant u}}\left(\prod_{t=1}^{m}\eta_{t-i_t}\right).$$

先看一个引理.

引理 $\forall\ j\in[0,\frac{K(K+1)}{2}]$

$$V^j(12\cdots(K-1))V^0_K+V^{j-1}(12\cdots(K-1))V^1_K+\cdots+V^1(12\cdots(K-1))V^{j-1}_K+V^0(12\cdots(K-1))V^j_K$$
$$=\sum_{\substack{(i_1,i_2,\cdots,i_K)\\ j=\sum_{u=1}^{K} i_u\\ 0\leqslant i_u\leqslant u}} V^{i_1}_1V^{i_2}_2\cdots V^{i_K}_K=V^j(12\cdots K).$$

证 $\because$

$$V^j(12\cdots K)=\sum_{\substack{(i_1,i_2,\cdots,i_K)\\ j=\sum_{u=1}^{K} i_u\\ 0\leqslant i_u\leqslant u}} V^{i_1}_1V^{i_2}_2\cdots V^{i_K}_K$$

$$=\sum_{\substack{(i_1,i_2,\cdots,i_{K-1})\\ j=\sum_{u=1}^{K-1} i_u\\ 0\leqslant i_u\leqslant u}} V^{i_1}_1V^{i_2}_2\cdots V^{i_{K-1}}_{K-1}V^0_K+\sum_{\substack{(i_1,i_2,\cdots,i_{K-1})\\ j-1=\sum_{u=1}^{K-1} i_u\\ 0\leqslant i_u\leqslant u}} V^{i_1}_1V^{i_2}_2\cdots V^{i_{K-1}}_{K-1}V^1_K+\sum_{\substack{(i_1,i_2,\cdots,i_{K-1})\\ j-2=\sum_{u=1}^{K-1} i_u\\ 0\leqslant i_u\leqslant u}} V^{i_1}_1V^{i_2}_2\cdots V^{i_{K-1}}_{K-1}V^2_K+\cdots$$

$$+\sum_{\substack{(i_1,i_2,\cdots,i_{K-1})\\ 1=\sum_{u=1}^{K-1} i_u\\ 0\leqslant i_u\leqslant u}} V^{i_1}_1V^{i_2}_2\cdots V^{i_{K-1}}_{K-1}V^{j-1}_K+\sum_{\substack{(i_1,i_2,\cdots,i_{K-1})\\ 0=\sum_{u=1}^{K-1} i_u\\ 0\leqslant i_u\leqslant u}} V^{i_1}_1V^{i_2}_2\cdots V^{i_{K-1}}_{K-1}V^j_K$$

$$=V^j(12\cdots(K-1))V^0_K+V^{j-1}(12\cdots(K-1))V^1_K+V^{j-2}(12\cdots(K-1))V^2_K+\cdots$$
$$+V^1(12\cdots(K-1))V^{j-1}_K+V^0(12\cdots(K-1))V^j_K.$$

$\therefore$ 引理成立.

定理 1 式(1)中的 $A_j(n)=V^j(12\cdots n)=\sum_{\substack{(i_1,i_2,\cdots,i_n)\\ j=\sum_{u=1}^{n} i_u\\ 0\leqslant i_u\leqslant u}}\left(\prod_{t=1}^{n}\eta_{t-i_t}\right)$，其中 $j\in[0,\frac{n(n+1)}{2}]$

证 $n=2$ 时，

$$\sum_{j=0}^{3}b^j\cdot A_j(2)=(\eta_1+b\eta_0)(\eta_2+b\eta_1+b^2\eta_0)=(1,b)\begin{pmatrix}\eta_1\\ \eta_0\end{pmatrix}(1,b,b^2)\begin{pmatrix}\eta_2\\ \eta_1\\ \eta_0\end{pmatrix}$$

$$=(1,b)\begin{pmatrix}V^0\eta_1\\ V^1\eta_1\end{pmatrix}(1,b,b^2)\begin{pmatrix}V^0\eta_2\\ V^1\eta_2\\ V^2\eta_2\end{pmatrix}=(1,b)\begin{pmatrix}V^0_1\\ V^1_1\end{pmatrix}(1,b,b^2)\begin{pmatrix}V^0_2\\ V^1_2\\ V^2_2\end{pmatrix}$$

$$=(1,b)\begin{pmatrix} V_1^0 & V_1^0 b & V_1^0 b^2 \\ V_1^1 & V_1^1 b & V_1^1 b^2\end{pmatrix}\begin{pmatrix}V_2^0\\V_2^1\\V_2^2\end{pmatrix} = (1,b)\begin{pmatrix}V_1^0V_2^0+V_1^0V_2^1b+V_1^0V_2^2b^2\\ V_1^1V_2^0+V_1^1V_2^1b+V_1^1V_2^2b^2\end{pmatrix}$$

$$=V_1^0V_2^0+b(V_1^0V_2^1+V_1^1V_2^0)+b^2(V_1^0V_2^2+V_1^1V_2^1)+b^3V_1^1V_2^2$$

$$=\sum_{\substack{(i_1,i_2)\\0=i_1+i_2\\0\leqslant i_1\leqslant 1\\0\leqslant i_2\leqslant 2}} V_1^{i_1}V_2^{i_2}+b\sum_{\substack{(i_1,i_2)\\1=i_1+i_2\\0\leqslant i_1\leqslant 1\\0\leqslant i_2\leqslant 2}} V_1^{i_1}V_2^{i_2}+b^2\sum_{\substack{(i_1,i_2)\\2=i_1+i_2\\0\leqslant i_1\leqslant 1\\0\leqslant i_2\leqslant 2}} V_1^{i_1}V_2^{i_2}+b^3\sum_{\substack{(i_1,i_2)\\3=i_1+i_2\\0\leqslant i_1\leqslant 1\\0\leqslant i_2\leqslant 2}} V_1^{i_1}V_2^{i_2}$$

$$=V^0(12)+b\cdot V^1(12)+b^2V^2(12)+b^3V^3(12)=\sum_{j=0}^{3}b^j\cdot V^j(12).$$

$$\Rightarrow A_j(2)=V^j(12);\qquad j\in[0,3],$$

∴　$n=2$ 时结论成立.

假设 $n=K-1$ 时结论成立,即 $j\in\left[0,\dfrac{(K-1)K}{2}\right]$,有 $A_j(K-1)=V^j(12\cdots(K-1))$.

下面来证 $n=K$ 时结论成立

$$\sum_{j=0}^{\frac{K(K+1)}{2}} b^j\cdot A_j(K)=(\eta_1+b\eta_0)(\eta_2+b\eta_1+b^2\eta_0)\cdots(\eta_{K-1}+b\eta_{K-2}+\cdots+b^{K-1}\eta_0)$$
$$(\eta_K+b\eta_{K-1}+\cdots+b^K\eta_0)$$

$$(\text{归纳假设})=\sum_{j=0}^{\frac{K(K-1)}{2}} b^j\cdot V^j(12\cdots(K-1))(V_K^0+bV_K^1+\cdots+b^KV_K^K)$$

$$=\sum_{j=0}^{\frac{K(K-1)}{2}} b^j\cdot V^j(12\cdots(K-1))\cdot V_K^0+\sum_{j=0}^{\frac{K(K-1)}{2}} b^{j+1}\cdot V^j(12\cdots(K-1))\cdot V_K^1+\cdots+$$
$$\sum_{j=0}^{\frac{K(K-1)}{2}} b^{j+K-1}V^j(12\cdots(K-1))V_K^{K-1}+\sum_{j=0}^{\frac{K(K-1)}{2}} b^{j+K}V^j(12\cdots(K-1))V_K^K$$

$$=V^0(12\cdots(K-1))V_K^0+b(V^1(12\cdots(K-1))V_K^0+V^0(12\cdots(K-1))V_K^1)+$$
$$b^2(V^2(12\cdots(K-1))V_K^0+V^1(12\cdots(K-1))V_K^1+V^0(12\cdots(K-1))V_K^2)+\cdots+b^{\frac{K(K+1)}{2}-1}(V^{\frac{K(K-1)}{2}}(12\cdots(K$$
$$-1))V_K^{K-1}+V^{\frac{K(K-1)}{2}-1}(12\cdots(K-1))V_K^K)+b^{\frac{K(K+1)}{2}}\cdot V^{\frac{K(K-1)}{2}}(12\cdots(K-1))V_K^K$$

$$(\text{引理})=V^0(12\cdots K)+bV^1(12\cdots K)+b^2V^2(12\cdots K)+\cdots+b^{\frac{K(K+1)}{2}-1}V^{\frac{K(K+1)}{2}-1}(12\cdots K)+$$
$$b^{\frac{K(K+1)}{2}}\cdot V^{\frac{K(K+1)}{2}}(12\cdots K)=\sum_{j=0}^{\frac{K(K+1)}{2}} b^j\cdot V^j(12\cdots K)$$

$$\Rightarrow\forall\ j\in\left[0,\frac{K(K+1)}{2}\right]\text{有 }A_j(K)=V^j(12\cdots K).$$

由归纳法知,结论成立.

2　算　例

下面我们通过求 $A_4(4)$ 来看所给定理 1 中的公式是如何快速计算的.

$$
\begin{aligned}
A_4(4) = V^4(1234) &= \sum_{\substack{(i_1,i_2,i_3,i_4) \\ 4=\sum_{u=1}^{4} i_u \\ 0\leqslant i_u \leqslant u}} V_1^{i_1}V_2^{i_2}V_3^{i_3}V_4^{i_4} \\
&= V_1^1V_2^1V_3^1V_4^1 + V_1^1V_2^1V_3^2V_4^0 + V_1^1V_2^1V_3^0V_4^2 + V_1^1V_2^2V_3^1V_4^0 + V_1^1V_2^2V_3^0V_4^1 + V_1^1V_2^0V_3^3V_4^0 + \\
&\quad V_1^1V_2^0V_3^2V_4^1 + V_1^1V_2^0V_3^1V_4^2 + V_1^1V_2^0V_3^0V_4^3 + V_1^0V_2^1V_3^1V_4^2 + V_1^0V_2^1V_3^3V_4^0 + V_1^0V_2^1V_3^2V_4^1 + \\
&\quad V_1^0V_2^1V_3^0V_4^3 + V_1^0V_2^2V_3^2V_4^0 + V_1^0V_2^2V_3^1V_4^1 + V_1^0V_2^2V_3^0V_4^2 + V_1^0V_2^0V_3^3V_4^1 + V_1^0V_2^0V_3^2V_4^2 + \\
&\quad V_1^0V_2^0V_3^1V_4^3 + V_1^0V_2^0V_3^0V_4^4 \qquad (2) \\
&= \eta_0\eta_1\eta_2\eta_3 + \eta_0\eta_1^2\eta_4 + \eta_0\eta_1\eta_3\eta_2 + \eta_0\eta_0\eta_2\eta_4 + \eta_0^2\eta_3^2 + \eta_0^2\eta_2\eta_4 + \eta_0\eta_2\eta_1\eta_3 + \eta_0\eta_2^2\eta_2 + \\
&\quad \eta_0\eta_2\eta_3\eta_1 + \eta_1^2\eta_2^2 + \eta_1^2\eta_0\eta_4 + \eta_1^3\eta_3 + \eta_1^2\eta_3\eta_1 + \eta_1\eta_0\eta_1\eta_4 + \eta_1\eta_0\eta_2\eta_3 + \eta_1\eta_0\eta_3\eta_2 + \eta_1\eta_2\eta_0\eta_3 \\
&\quad + \eta_1\eta_2\eta_1\eta_2 + \eta_1\eta_2\eta_2\eta_1 + \eta_1\eta_2\eta_3\eta_0 \\
&= 8\eta_0\eta_1\eta_2\eta_3 + 3\eta_0\eta_1^2\eta_4 + 2\eta_0^2\eta_2\eta_4 + \eta_0^2\eta_3^2 + \eta_0\eta_2^3 + 2\eta_1^3\eta_3 + 3\eta_1^2\eta_2^2.
\end{aligned}
$$

3　结束语

尽管有定理1使我们能快速算出$A_j(n)$，但我们认为，对于对称的$A_{\frac{n(n+1)}{2}-j}(n)$有更简洁的办法. 即

定理2　$\forall\, j \in [0,[\frac{n(n+1)}{2}]]$时，有$A_{\frac{n(n+1)}{2}-j}(n) = \sum\limits_{(j_1,j_2,\cdots,j_n)} \eta_{j_1}\eta_{j_2}\cdots\eta_{j_n}$，其中求和的$(j_1, j_2,\cdots,j_n)$恰好就是$A_j(n) = \sum\limits_{\substack{(i_1,i_2,\cdots,i_n) \\ j=\sum_{u=1}^{n} i_u \\ 0\leqslant i_u\leqslant u}} V_1^{i_1}V_2^{i_2}\cdots V_n^{i_n}$中所有的$(i_1,i_2,\cdots,i_n)$.

证　
$$
\begin{aligned}
A_{\frac{n(n+1)}{2}-j}(n) &= \sum_{(j_1,j_2,\cdots,j_n)} \eta_{j_1}\eta_{j_2}\cdots\eta_{j_n} \\
&= \sum_{\substack{(i_1,i_2,\cdots,i_n) \\ \frac{n(n+1)}{2}-j=\sum_{u=1}^{n} i_u \\ 0\leqslant i_u\leqslant u}} \eta_{1-i_1}\eta_{2-i_2}\cdots\eta_{n-i_n} = \sum_{\substack{(i_1,i_2,\cdots,i_n) \\ j=\sum_{u=1}^{n}(u-i_u) \\ 0\leqslant i_u\leqslant u}} \left(\prod_{u=1}^{n}\eta_{u-i_u}\right).
\end{aligned}
$$

$\therefore\ j_u = u - i_u. \qquad (u = 1,2,\cdots,n)$

$\Rightarrow (i_1,i_2,\cdots,i_n)$与$(j_1,j_2,\cdots,j_n)$是1－1对称的.　　（毕）

如求$A_6(4)$时，只须由式(2)即得

$$
\begin{aligned}
A_6(4) &= \eta_1\eta_1\eta_1\eta_1 + \eta_1\eta_1\eta_2\eta_0 + \eta_1\eta_1\eta_0\eta_2 + \eta_1\eta_2\eta_1\eta_0 + \eta_1\eta_2\eta_0\eta_4 + \eta_1\eta_0\eta_3\eta_0 + \eta_1\eta_0\eta_2\eta_1 + \eta_1\eta_0\eta_1\eta_2 \\
&\quad + \eta_1\eta_0\eta_0\eta_3 + \eta_0\eta_1\eta_1\eta_2 + \eta_0\eta_1\eta_3\eta_0 + \eta_0\eta_1\eta_2\eta_1 + \eta_0\eta_1\eta_0\eta_3 + \eta_0\eta_2\eta_2\eta_0 + \eta_0\eta_2\eta_1\eta_1 + \\
&\quad \eta_0\eta_2\eta_0\eta_2 + \eta_0\eta_0\eta_3\eta_1 + \eta_0\eta_0\eta_2\eta_2 + \eta_0\eta_0\eta_1\eta_3 + \eta_0\eta_0\eta_0\eta_4 \\
&= \eta_1^4 + 9\eta_0\eta_1^2\eta_2 + 6\eta_0^2\eta_1\eta_3 + 3\eta_0^2\eta_2^2 + \eta_0^3\eta_4.
\end{aligned}
$$

参 考 文 献

1　罗乔林. 一个组合数学问题. 曲阜师范大学学报，1994，(1)：13～18

On A Combinatorial Mathematics Problem

Zhou Xuesong
(Department of Basic Courses)

Abstract In[1], the expansion formula of $(\eta_1+b\eta_0)(\eta_2+b\eta_1+b^2\eta_0)\cdots(\eta_n+b\eta_{n-1}+\cdots+b^n\eta_0)=\sum_{j=0}^{C_2^{n+1}} b^j\cdot A_j(n)$ is disscussed. In this paper, the same problem is discussed and a concise formula for $A_j(n)$ is given. That is

$$A_j(n)=\sum_{\substack{(i_1,i_2,\cdots,i_n)\\ j=\sum_{s=1}^{n} i_s\\ 0\leq i_s\leq s}}\left(\prod_{t=1}^{n}\eta_{t-i_t}\right).$$

Key words Combinatorial theory; polynomial expansion; Operator

(上接第 61 页)

On Code Constructions for Computer Fault - tolerant Byte Organized Memory Subsystem

Wang Bingrong

(Department of Electrical Engineering)

Abstract A computer fault-tolerant byte organized subsystem which based on the byte error-correcting codes is presented; two kinds of byte unidirectional error-correcting codes used for fault-tolerant are constructed. Compare to the other byte error-correcting codes such as RS codes, the constructed codes have higher code efficiency and code rate. They can be easily realized by software or hardware, or integrated in the memory devices.

Key words Code/Decode; Byte unidrectional error-correcting code; Byte bidirectional error-correcting code; Fault-tolerant